AD-A058 818 DELAWARE UNIV NEWARK DEPT OF MECHANICAL AND AEROSPA--ETC F/G 13/5
AN ANALYTICAL MODEL FOR BONDED JOINT ANALYSIS IN COMPOSITE STRU--ETC(U)
FEB 77 R C WETHERHOLD , J R VINSON AFOSR-74-2739

UNCLASSIFIED MAE-TR-198 AFOSR-TR-78-1337 NL

1 OF 2
AD
A058 818

AFOSR SCIENTIFIC REPORT

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MAE-TR-198

LEVEL II

AN ANALYTICAL MODEL FOR BONDED JOINT ANALYSIS IN COMPOSITE STRUCTURES INCLUDING HYGROTHERMAL EFFECTS.

Interim rept.,

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AND

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2307 B1

FEBRUARY 1977

139p.

AIR FORCE OFFICE OF SCIENTIFIC RESEARCH
UNITED STATES AIR FORCE
CONTRACT NUMBER AFOSR-74-2739

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REPORT DOCUMENTATION PAGE

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1. REPORT NUMBER: AFOSR-TR- 78-1337
2. GOVT ACCESSION NO.:
3. RECIPIENT'S CATALOG NUMBER:
4. TITLE (and Subtitle): AN ANALYTICAL MODEL FOR BONDED JOINT ANALYSIS IN COMPOSITE STRUCTURES INCLUDING HYGROTHERMAL EFFECTS
5. TYPE OF REPORT & PERIOD COVERED: INTERIM
6. PERFORMING ORG. REPORT NUMBER:
7. AUTHOR(s): ROBERT C WETHERHOLD, JACK R VINSON
8. CONTRACT OR GRANT NUMBER(s): AFOSR 74-2739
9. PERFORMING ORGANIZATION NAME AND ADDRESS: UNIVERSITY OF DELAWARE, MECHANICAL & AEROSPACE ENGINEERING DEPARTMENT, NEWARK, DELAWARE 19711
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 2307B1, 61102F
11. CONTROLLING OFFICE NAME AND ADDRESS: AIR FORCE OFFICE OF SCIENTIFIC RESEARCH/NA, BLDG 410, BOLLING AIR FORCE BASE, D C 20332
12. REPORT DATE: Feb 77
13. NUMBER OF PAGES: 138
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office):
15. SECURITY CLASS. (of this report): UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE:

16. DISTRIBUTION STATEMENT (of this Report):

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17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report):
18. SUPPLEMENTARY NOTES:
19. KEY WORDS (Continue on reverse side if necessary and identify by block number):

COMPOSITE MATERIALS
BONDED ADHESIVE JOINTS
LAMINATED STRUCTURES

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A plane-strain model is developed for symmetric, anisotropic laminated plates subjected to thermal, moisture, and mechanical loading including surface tractions. Provision is made to allow assessment of the relative importance of the normal strain and of various off-axis effects by the use of tracing constants. The solution for stresses and displacements is given for a single plate element, and the computations are demonstrated by numerical examples. The basic equations for a bonded joint composed of plane-strain plate adherends are derived and discussed. References to alternate derivations are given whenever appropriate in order to facilitate comparisons.

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(2)

AN ANALYTICAL MODEL FOR BONDED JOINT ANALYSIS IN COMPOSITE STRUCTURES INCLUDING HYGROTHERMAL EFFECTS*

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February 1977

* Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, Grant Number AFOSR-74-2739. This document has been approved for public release; distribution is unlimited.

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ABSTRACT

A plane-strain model is developed for symmetric, anisotropic laminated plates subjected to thermal, moisture, and mechanical loadings including surface tractions. Provision is made to allow assessment of the relative importance of the normal strain and of various off-axis effects by the use of tracing constants. The solution for stresses and displacements is given for a single plate element, and the computations are demonstrated by numerical examples. The basic equations for a bonded joint composed of plane-strain plate adherends are derived and discussed. References to alternate derivations are given whenever appropriate in order to facilitate comparisons.

NOMENCLATURE

h	=	total thickness of the laminate.
H_{K+1}	=	z distance from the geometrical mid-plane of the laminate to the inter-face between lamina (K) and lamina (K+1).
k subscript		denotes the k^{th} lamina.
m	=	percent of moisture content measured from some stress free condition; a smooth function of x and z.
m superscript		denotes moisture dependent term.
Q_{ij}	=	terms of the lamina stiffness matrix referred to the principal material axes.
$\bar{Q}_{ij}$	=	terms of the lamina stiffness matrix referred to an arbitrary set of axes.
S_{ij}	=	terms of the lamina compliance matrix referred to the principal material coordinate system.
$\bar{S}_{ij}$	=	terms of the lamina compliance matrix referred to an arbitrary set of axes.
$[T]$	=	transformation matrix used between the material principal coordinates and the geometric coordinates.

T Superscript		denotes temperature dependent term.
u°	=	the laminate mid-plane longitudinal displacement in the x direction.
w°	=	the laminate mid-plane transverse displacement in the z direction.
z	=	coordinate in the thickness direction, measured from the geometric midsurface.
$\alpha_1, \alpha_2, \alpha_3$	=	coefficients of thermal expansion, along principal material axes.
$\alpha_x, \alpha_y, \alpha_z, \alpha_{xy}$	=	coefficients of thermal expansion with respect to laminate coordinate system.
$\bar{\alpha}_K$	=	boundary constant for continuity of σ_z between laminae.
$\beta_1, \beta_2, \beta_3$	=	coefficients of moisture or hygroscopic expansion, in material principal axes.
$\beta_x, \beta_y, \beta_z, \beta_{xy}$	=	coefficients of moisture or hygroscopic expansion with respect to laminate coordinate system.
ΔT	=	temperature measured from some stress free condition; a smooth function of x and z .
ε_i	=	normal dilitational strains
ε_{ij}	=	shear strains, $(i \neq j)$.
σ_i	=	normal stresses.
τ_{ij}	=	shear stresses, $(i \neq j)$.

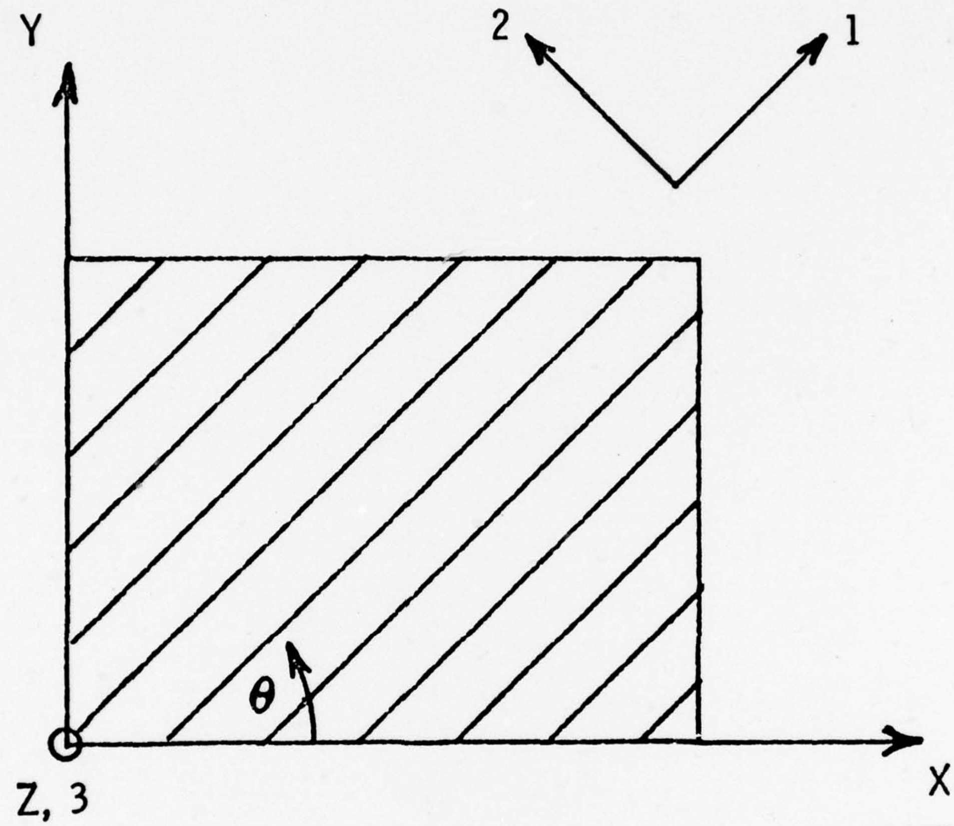

FIGURE I

Specimen and Fiber Coordinate System

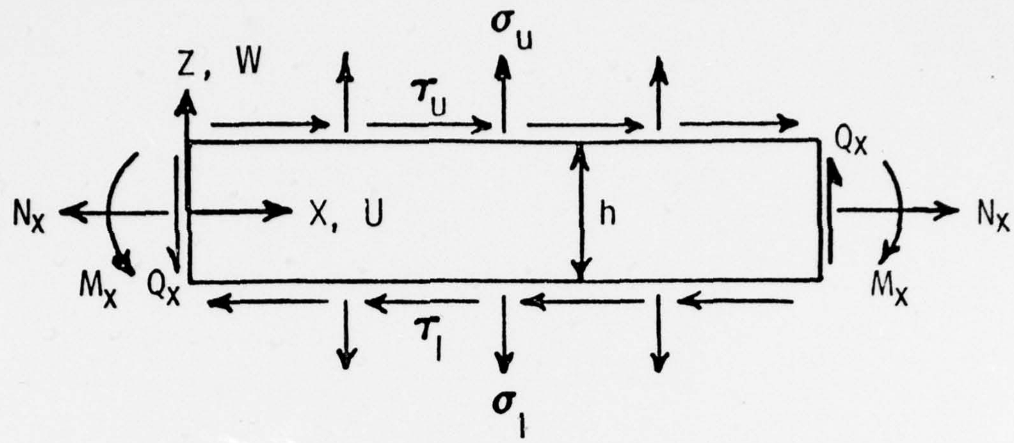

FIGURE 2

Integrated Stress Resultant and
Surface Traction Definitions

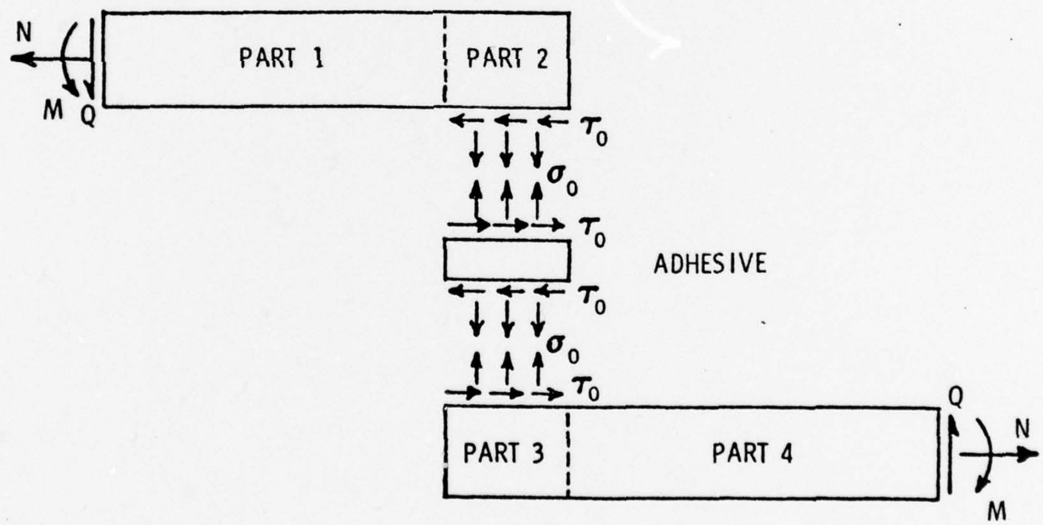

FIGURE 3

Four Element Single Lap Joint

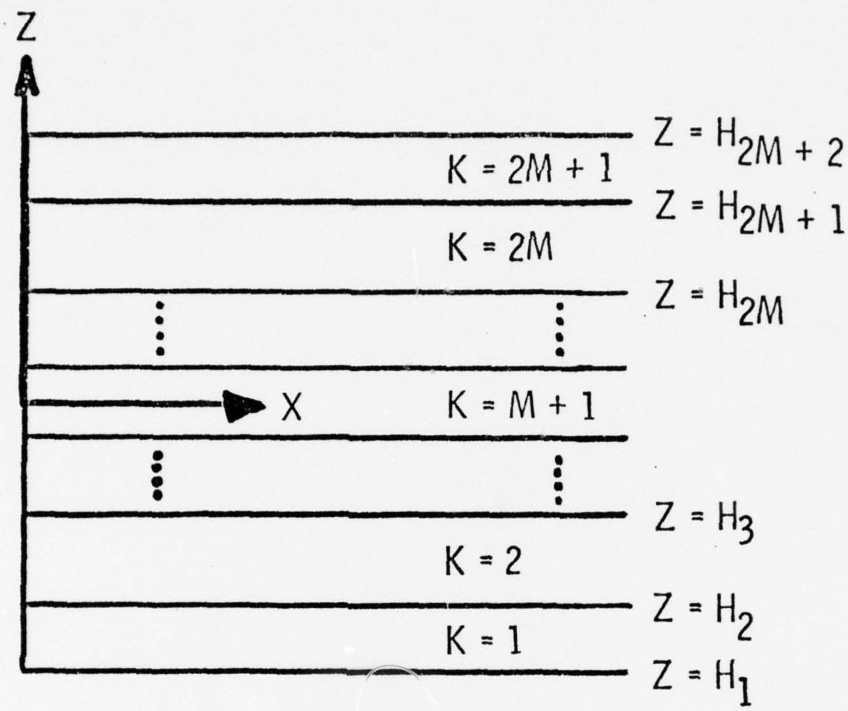

FIGURE 4

Lamina and Height Definitions for a Symmetric Plate with an Odd Number of Laminae

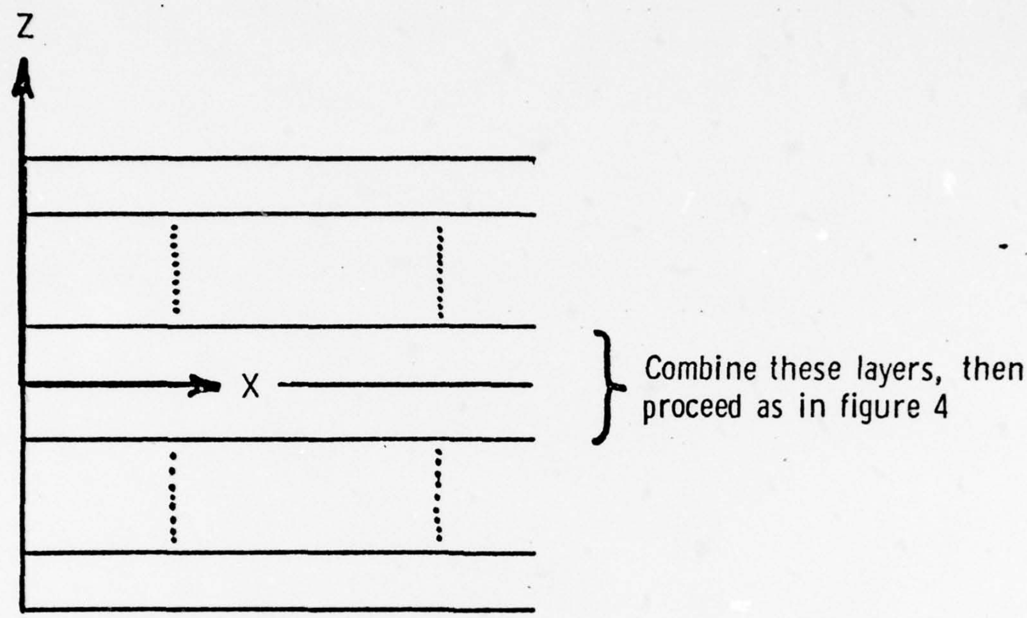

FIGURE 5

Symmetric Plate, Even Number of Laminae

CHAPTER 1

Introduction and Summary of Basic Assumptions

In many advanced structural applications, the advantages of high strength-to weight fibrous composites are well-proven. What is often lacking, however, is a suitable method of analysis for laminated composite plates bonded into a structure and subjected to a full range of mechanical, thermal, and moisture loadings. The usual plane stress plate theory ($\tau_{xz} = \sigma_z = \tau_{yz} = 0$) is not suitable because of its failure to account for the surface tractions produced by bonding. The theory developed herein is based on a plane-strain approach, which would be suitable for many panel constructions. [Ref. 1,2,3]

If the composite is polymeric, thermal and moisture effects (termed "hygroscopic") become important. The magnitude of the induced strains due to these effects as well as material property degradation at elevated temperature demand their inclusion in the analysis. A general procedure has thus been developed to predict stresses, displacements, and strains for the symmetric laminated plate under combined thermal, moisture, and mechanical loadings.

In order to obtain a solution to the laminated plate problem, certain assumptions have been found useful. These assumptions are summarized by the following:

1) A plane-strain condition is assumed in the y direction, and y dependency of all displacements and of all stresses is assumed zero. (See Figure 1) The three dimensional elasticity problem is thus reduced to two dimensions.

2) The laminate is assumed symmetric with respect to its midplane; however, either an odd or an even number of layers may be accomodated. (See Figures 4 and 5).

3) The material coordinate system (1,2,3) is a Cartesian system with 3 mutually orthogonal planes of symmetry.

4) The transverse shear stress distribution in the laminate may be represented by a quadratic function in z times a function of x . (See Figure 6).

5) The theory presented here is linear; the results of in-plane loadings, surface tractions, and thermal and moisture loadings may therefore be superimposed.

6) The adhesive between the plate elements is subject to shear and normal stresses only as shown in Figures 3 and 16.

7) Both transverse shear strain, ε_{xz}, and normal strain, ε_z, are included. Provision is made to neglect the normal strain, if desired, by use of a tracing constant.

8) The given temperature and moisture functions, ΔT and m, will be assumed smooth (possessing continuous derivatives) in x, y and z throughout the laminate, and also will be of polynomial form in z.

9) All material properties are assumed constant. Should high temperature or moisture alter the properties significantly, average properties will be used.

CHAPTER 2

Derivation of the Governing Equations for a Plane-Strain Plate Element

2.1 Basic Elastic Relations and Discussion of the Plane-Strain Condition

Given a constant coefficient of thermal expansion, α, and a constant coefficient of moisture hygroscopic expansion, β, the constitutive relations may be written for the k^{th} lamina in the material principal coordinates:

$$\begin{bmatrix} \varepsilon_1-\alpha_1\Delta T-\beta_1 m \\ \varepsilon_2-\alpha_2\Delta T-\beta_2 m \\ \varepsilon_3-\alpha_3\Delta T-\beta_3 m \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{bmatrix}_k = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix}_k \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix}_k \quad (2.1.1)$$

where the ε_{ij} are components of the strain tensor. Note that the thermal and moisture expansion effects are purely dilatational in the material coordinate system. The terms of the above flexibility matrix [S] are given by the follow-

ing definitions: (Please note the definition of S_{44}, S_{55} and S_{66}.)

$$
\begin{aligned}
S_{11} &= 1/E_{11} \\
S_{22} &= 1/E_{22} \\
S_{33} &= 1/E_{33} \\
S_{12} &= -\nu_{21}/E_{22} \\
S_{13} &= -\nu_{31}/E_{33} \\
S_{23} &= -\nu_{32}/E_{33} \\
S_{44} &= 1/2G_{23} \\
S_{55} &= 1/2G_{13} \\
S_{66} &= 1/2G_{12}
\end{aligned}
\qquad (2.1.2)
$$

The symmetry of the [S] matrix implies that

$$\frac{\nu_{ij}}{E_{ii}} = \frac{\nu_{ij}}{E_{jj}} \qquad (2.1.3)$$

so that there are nine independent constants for each lamina. This corresponds to two (or, automatically, three) orthogonal planes of symmetry in the material coordinate system of the lamina. [Ref. 4]

If we form the inverse of [S],

$$[Q]_k = [S]_k^{-1} \quad , \tag{2.1.4}$$

we have

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix}_k = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & Q_{23} & 0 & 0 & 0 \\ Q_{13} & Q_{23} & Q_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix}_k \begin{bmatrix} \varepsilon_1-\alpha_1\Delta T-\beta_1 m \\ \varepsilon_2-\alpha_2\Delta T-\beta_2 m \\ \varepsilon_3-\alpha_3\Delta T-\beta_3 m \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{bmatrix}_k \tag{2.1.5}$$

where

$$\begin{aligned} Q_{11} &= (1-\nu_{23}\nu_{32})E_{11}/DET \\ Q_{22} &= (1-\nu_{13}\nu_{31})E_{22}/DET \\ Q_{33} &= (1-\nu_{12}\nu_{21})E_{33}/DET \\ Q_{12} &= (\nu_{21} + \nu_{23}\nu_{31})E_{11}/DET \\ Q_{13} &= (\nu_{31} + \nu_{32}\nu_{21})E_{11}/DET \\ Q_{23} &= (\nu_{32} + \nu_{31}\nu_{12})E_{22}/DET \end{aligned} \tag{2.1.6}$$

$$Q_{44} = 2G_{23}$$

$$Q_{55} = 2G_{13}$$

$$Q_{66} = 2G_{12}$$

where

$$DET = (1-\nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{12}\nu_{23}\nu_{31})$$

Having defined the [S] matrix and its inverse, [Q], in material coordinates, we now turn to writing the constitutive relations in the geometric system. The stress and strain tensors may be transformed into the geometric coordinates (x,y,z) by use of the transformation matrix $[T]_k$ to produce a simple rotation about the 3 (z) axis. (See Figure 1)

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix}_k = [T]_k \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{bmatrix}_k \qquad (2.1.7)$$

$$\begin{bmatrix} \varepsilon_1-\alpha_1\Delta T-\beta_1 m \\ \varepsilon_2-\alpha_2\Delta T-\beta_2 m \\ \varepsilon_3-\alpha_3\Delta T-\beta_3 m \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{bmatrix}_k = [T]_k \begin{bmatrix} \varepsilon_x-\alpha_x\Delta T-\beta_x m \\ \varepsilon_y-\alpha_y\Delta T-\beta_y m \\ \varepsilon_z-\alpha_z\Delta T-\beta_z m \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy}-\alpha_{xy}\Delta T-\beta_{xy} m \end{bmatrix}_k \tag{2.1.8}$$

The transformation matrix is given by

$$[T]_k = \begin{bmatrix} \bar{M}^2 & \bar{N}^2 & 0 & 0 & 0 & 2\bar{M}\bar{N} \\ \bar{N}^2 & \bar{M}^2 & 0 & 0 & 0 & -2\bar{M}\bar{N} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{M} & -\bar{N} & 0 \\ 0 & 0 & 0 & \bar{N} & \bar{M} & 0 \\ -\bar{M}\bar{N} & \bar{M}\bar{N} & 0 & 0 & 0 & \bar{M}^2-\bar{N}^2 \end{bmatrix} \tag{2.1.9}$$

where $\bar{M} = \cos\theta$ and $\bar{N} = \sin\theta$ (see Figure 1)

Also, conveniently, the inverse of [T] is given by

$$[T]_k^{-1} = \begin{bmatrix} \bar{M}^2 & \bar{N}^2 & 0 & 0 & 0 & -2\bar{M}\bar{N} \\ \bar{N}^2 & \bar{M}^2 & 0 & 0 & 0 & 2\bar{M}\bar{N} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{M} & \bar{N} & 0 \\ 0 & 0 & 0 & -\bar{N} & \bar{M} & 0 \\ \bar{M}\bar{N} & -\bar{M}\bar{N} & 0 & 0 & 0 & \bar{M}^2-\bar{N}^2 \end{bmatrix} \tag{2.1.10}$$

which is to say that

$$[T(\theta)]_k^{-1} = [T(-\theta)]_k \qquad (2.1.11)$$

Thus, we may write:

$$\begin{aligned}
\alpha_x &= M^2\alpha_1 + N^2\alpha_2 & \beta_x &= M^2\beta_1 + N^2\beta_2 \\
\alpha_y &= N^2\alpha_1 + M^2\alpha_2 & \beta_y &= N^2\beta_1 + M^2\beta_2 \\
\alpha_z &= \alpha_3 & \beta_z &= \beta_3 \\
\alpha_{xy} &= MN(\alpha_1-\alpha_2) & \beta_{xy} &= MN(\beta_1-\beta_2)
\end{aligned} \qquad (2.1.12)$$

By combining equations (2.1.5), (2.1.7), and (2.1.8) we have the constitutive relations in the geometrical system:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{13} & 0 & 0 & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{23} & 0 & 0 & \bar{Q}_{26} \\ \bar{Q}_{13} & \bar{Q}_{23} & \bar{Q}_{33} & 0 & 0 & \bar{Q}_{36} \\ 0 & 0 & 0 & \bar{Q}_{44} & \bar{Q}_{45} & 0 \\ 0 & 0 & 0 & \bar{Q}_{45} & \bar{Q}_{55} & 0 \\ \bar{Q}_{61} & \bar{Q}_{62} & \bar{Q}_{63} & 0 & 0 & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \varepsilon_x-\alpha_x\Delta T-\beta_x m \\ \varepsilon_y-\alpha_y\Delta T-\beta_y m \\ \varepsilon_z-\alpha_z\Delta T-\beta_z m \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy}-\alpha_{xy}\Delta T-\beta_{xy} m \end{bmatrix}_k$$

(2.1.13)

where the entries of $[\bar{Q}]_k$ are given by

$$[\bar{Q}]_k = [T]_k^{-1} \ [Q]_k \ [T]_k \tag{2.1.14}$$

and the form of $[\bar{Q}]$ is that of a system with one plane of elastic symmetry.

We may also invert (2.1.13), noting that an $[\bar{S}]$ matrix is consistently defined by either

$$[\bar{S}]_k = [\bar{Q}]_k^{-1} \tag{2.1.15}$$

or

$$[\bar{S}]_k = [T]_k^{-1} \ [S]_k \ [T]_k \tag{2.1.16}$$

With the use of the true elastic strain quantities, the $[\bar{Q}]$ and $[\bar{S}]$ matrices are not symmetric. The non-symmetry is in the terms:

$$\bar{Q}_{16} = 2\bar{Q}_{61}$$

$$\bar{Q}_{26} = 2\bar{Q}_{62}$$

$$\bar{Q}_{36} = 2\bar{Q}_{63}$$

$$\bar{S}_{16} = 2\bar{S}_{61}$$

$$\bar{S}_{26} = 2\bar{S}_{62}$$

$$\bar{S}_{36} = 2\bar{S}_{63}$$

If we had chosen to use the "engineering" shear strains, i.e. if the strain quantities used had been

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix}_k$$

Then the $\bar{Q}$ and $\bar{S}$ matrices would have been symmetric.

Returning to the elasticity strains used herein, we proceed to invert equation (2.1.13) and introduce our plane-strain assumption:

$$\begin{bmatrix} \epsilon_x - \alpha_x \Delta T - \beta_x m \\ \cancelto{0}{\epsilon_y} - \alpha_y \Delta T - \beta_y m \\ \epsilon_z - \alpha_z \Delta T - \beta_z m \\ \cancelto{0}{\epsilon_{yz}} \\ \epsilon_{xz} \\ \cancelto{0}{\epsilon_{xy}} - \alpha_{xy} \Delta T - \beta_{xy} m \end{bmatrix}_k = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{13} & 0 & 0 & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{23} & 0 & 0 & \bar{S}_{26} \\ \bar{S}_{13} & \bar{S}_{23} & \bar{S}_{33} & 0 & 0 & \bar{S}_{36} \\ 0 & 0 & 0 & \bar{S}_{44} & \bar{S}_{45} & 0 \\ 0 & 0 & 0 & \bar{S}_{45} & \bar{S}_{55} & 0 \\ \bar{S}_{61} & \bar{S}_{62} & \bar{S}_{63} & 0 & 0 & \bar{S}_{66} \end{bmatrix}_k \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix}_k \tag{2.1.17}$$

where the $\bar{S}_{ij}$ are given by either (2.1.15) or (2.1.16)

Defining the displacements,

u = displacement in x direction

v = displacement in y direction

w = displacement in z direction

we specify, consistent with the plane-strain assumption,

$$\frac{\partial w}{\partial y} = 0 \qquad w = w(x,z) \tag{2.1.18}$$

$$\frac{\partial u}{\partial y} = 0 \qquad u = u(x,z) \tag{2.1.19}$$

Thus, we write the strain-displacement relations for the plane-strain condition:

$$\varepsilon_x = \frac{\partial u}{\partial x} \tag{2.1.20}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \tag{2.1.21}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \neq 0 \tag{2.1.22}$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \cancelto{0}{\frac{\partial w}{\partial y}}\right) = 0 \tag{2.1.23}$$

$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \tag{2.1.24}$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\cancelto{0}{\frac{\partial u}{\partial y}} + \frac{\partial v}{\partial x}\right) = 0 \tag{2.1.25}$$

Equation (2.1.22) includes the effect of transverse normal strain.

2.2 Equilibrium Equations

Care must be exercised due to the derivative nature of these equations. These equations are written for a continuum, so they will be subscripted for the k^{th} lamina to emphasize that the derivatives may not exist at the interface between laminae. No y variation is permitted in stresses, $(\frac{\partial()}{\partial y} = 0)$, for the plane strain condition. Thus, the equations become:

$$\frac{\partial\sigma_x}{\partial x}\bigg|_k + \frac{\partial\tau_{xz}}{\partial z}\bigg|_k = 0 \tag{2.2.1}$$

$$\frac{\partial\tau_{yx}}{\partial x}\bigg|_k + \frac{\partial\tau_{yz}}{\partial z}\bigg|_k = 0 \tag{2.2.2}$$

$$\frac{\partial\tau_{zx}}{\partial x}\bigg|_k + \frac{\partial\sigma_z}{\partial z}\bigg|_k = 0 \tag{2.2.3}$$

Note that if the material is specially orthotropic, i.e. if the geometric and principal material axes coincide so that $\theta = 0°$ or $90°$ in Figure 1, then equation (2.2.2) is made trivial by the proportional relation between τ_{xy}, τ_{yz} and the strain quantities ε_{xy}, ε_{yz} which are assumed zero.

2.3 Presumed Form for Transverse Shear Stress τ_{xz}; Derivation of Normal Stress σ_z

The form selected for the transverse shear stress, τ_{xz}, is that of a piecewise smooth quadratic in "z" times an unknown function in "x". Linear superposition is assumed to apply, i.e. the effects of upper and lower surface shears may be added in linearly. In what follows, subscript "k" will refer to the k^{th} lamina, and M is the total number of complete laminae below the midplane. For the laminate shown in Figure 6, M=2. The surface shear stress on the upper and lower surfaces are τ_u and τ_L, respectively, and ϕ is an unknown function of x only.

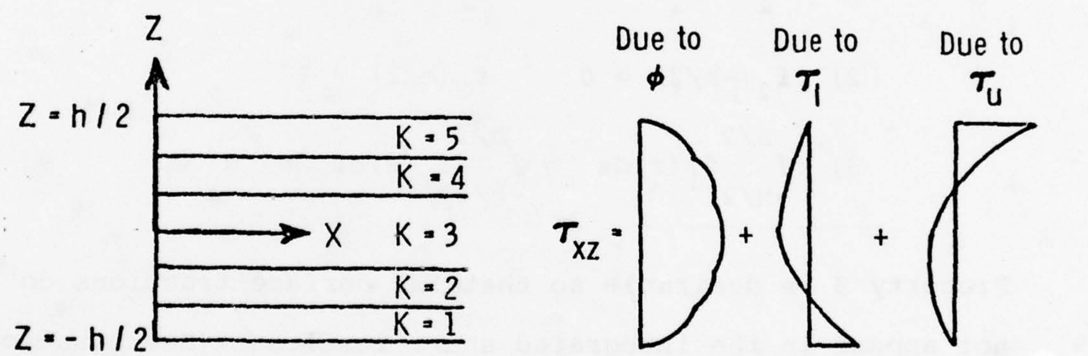

FIGURE 6

Assumed Form for Transverse Shear Stress, τ_{xz}

The assumed form for τ_{xz} will be continuously differentiable in x and piecewise differentiable in z:

$$\tau_{xz}(x,z)_K = \tau_L(x) f_1(z) + \tau_u(x) f_2(z) + \left[\bar{Q}_{55}\big|_K f_3(z) + b_{55}\big|_K\right] \phi(x) \quad (2.3.1)$$

where

$$f_1(z) = \left(-\frac{1}{4} - \frac{z}{h} + \frac{3z^2}{h^2}\right) \quad (2.3.2)$$

$$f_2(z) = \left(-\frac{1}{4} + \frac{z}{h} + \frac{3z^2}{h^2}\right) \quad (2.3.3)$$

$$f_3(z) = \left(1 - \frac{4z^2}{h^2}\right) \quad (2.3.4)$$

b_{55} is a constant which insures continuity of τ_{xz}. Functions f_1 and f_2 have been selected because of the following properties (See Ref. 5, p. 21);

1) $f_1(-h/2) = 1 \qquad f_1(h/2) = 0$

2) $f_2(-h/2) = 0 \qquad f_2(h/2) = 1$

3) $\int_{-h/2}^{h/2} f_1(z)dz = \int_{-h/2}^{h/2} f_2(z)dz = 0$

Property 3 is desirable so that the surface tractions do not appear in the integrated shear resultant. (See equation 2.5.31)

By definition, we must have

$$\tau_{xz}\big|_1 \left(-\frac{h}{2}\right) = \tau_L(x) \quad (2.3.5)$$

$$\tau_{xz}\big|_{2M+1} \left(\frac{h}{2}\right) = \tau_u(x) \quad (2.3.6)$$

We proceed to solve for the constants $b_{55|_k}$. The continuity of τ_{xz} through the laminate gives:

$$\tau_{xz|_{k+1}}(H_{k+1}) = \tau_{xz|_k}(H_{k+1}) \tag{2.3.7}$$

Using (2.3.1) with (2.3.5) we see that

$$b_{55|_1} = 0 \tag{2.3.8a}$$

Furthermore, the general condition (2.3.7) will reduce to

$$[\bar{Q}_{55|_k} - \bar{Q}_{55|_{k+1}}] f_3(H_{k+1}) + b_{55|_k} = b_{55|_{k+1}}$$

Thus a general expression for $k = 1 \rightarrow M$ is

$$b_{55|_{k+1}} = \sum_{r=1}^{k} f_3(H_{r+1}) \; [\bar{Q}_{55|_r} - \bar{Q}_{55|_{r+1}}] \tag{2.3.8b}$$

and there is reflective symmetry about the midsurface due to symmetric material and geometric properties.

$$\text{for } k = M + 2 \rightarrow 2M + 1$$

$$b_{55|_k} = b_{55|_{2M+2-k}} \tag{2.3.9}$$

Now that we have an expression for $\tau_{xz|_k}$, we turn to computation of $\sigma_{z|_k}$. Using equilibrium equation (2.2.3), and integrating with respect to z, we have

$$\sigma_z(x,z)_k = -\int^{z} \frac{\partial \tau_{xz}|_k}{\partial x}\, dz + d_k(x) \tag{2.3.10}$$

where $d_k(x)$ is a function of integration to be determined and $\int^z$ is an indefinite integral. Using our formula for $\tau_{xz}|_k$,

$$\sigma_z|_k = -\frac{\partial}{\partial x}\int^z \tau_{xz}|_k \, dz + d_k(x)$$

$$\sigma_z|_k = -\frac{\partial}{\partial x}\int^z \Big\{ \tau_L\left(-\frac{1}{4}-\frac{z}{h}+\frac{3z^2}{h^2}\right)+\tau_u\left(-\frac{1}{4}+\frac{z}{h}+\frac{3z^2}{h^2}\right) + \left[\bar{Q}_{55}|_k\left(1-\frac{4z^2}{h^2}\right)+b_{55}|_k\right]\phi\Big\}dz + d_k(x)$$

$$\sigma_z|_k = -\frac{d\tau_L}{dx}\left(-\frac{z}{4}-\frac{z^2}{2h}+\frac{z^3}{h^2}\right)-\frac{d\tau_u}{dx}\left(-\frac{z}{4}+\frac{z^2}{2h}+\frac{z^3}{h^2}\right) - \bar{Q}_{55}|_k\left(z-\frac{4z^3}{3h^2}\right)\frac{d\phi}{dx} - b_{55}|_k\, z\,\frac{d\phi}{dx} + d_k(x) \tag{2.3.11}$$

These functions of integration, d_k, depend on the surface tractions and on continuity of σ_z:

$$\sigma_z|_k(x,H_{k+1}) = \sigma_z|_{k+1}(x,H_{k+1}) \tag{2.3.12}$$

$$\sigma_z\left(x,-\tfrac{h}{2}\right)_1 = \sigma_L = -\frac{d\tau_L}{dx}\left[-\frac{1}{4}\left(\frac{-h}{2}\right)-\frac{1}{2h}\left(\frac{-h}{2}\right)^2+\frac{1}{h^2}\left(\frac{-h}{2}\right)^3\right] - \frac{d\tau_u}{dx}\left[-\frac{1}{4}\left(\frac{-h}{2}\right)+\frac{1}{2h}\left(\frac{-h}{2}\right)^2+\frac{1}{h^2}\left(\frac{-h}{2}\right)^3\right] - \bar{Q}_{55}|_1\left(\frac{-h}{2}-\frac{4}{3h^2}\left(\frac{-h}{2}\right)^3\right)\frac{d\phi}{dx} - 0 + d_1(x)$$

$$d_1(x) = -\frac{h}{8}\frac{d\tau_L}{dx}+\frac{h}{8}\frac{d\tau_u}{dx}-\frac{h}{3}\bar{Q}_{55}|_1\frac{d\phi}{dx}+\sigma_L$$

Using the continuity condition (2.3.12) for the first interface:

$$\sigma_z(x,H_2)_1 = \sigma_z(x,H_2)_2$$

or

$$\cancel{-\frac{d\tau_L}{dx}\left(-\frac{H_2}{4}-\frac{H_2^2}{2h}+\frac{H_2^3}{h^2}\right)} - \cancel{\frac{d\tau_u}{dx}\left(-\frac{H_2}{4}+\frac{H_2^2}{2h}+\frac{H_2^3}{h^2}\right)}$$

$$-\bar{Q}_{55}\big|_1\left(H_2-\frac{4H_2^3}{3h^2}\right)\frac{d\phi}{dx} - \cancelto{0}{b_{55}\big|_1} H_2\frac{d\phi}{dx} + d_1(x)$$

$$= \cancel{-\frac{d\tau_L}{dx}\left(-\frac{H_2}{4}-\frac{H_2^2}{2h}+\frac{H_2^3}{h^2}\right)} - \cancel{\frac{d\tau_u}{dx}\left(\frac{H_2}{4}+\frac{H_2^2}{2h}+\frac{H_2^3}{h^2}\right)}$$

$$-\bar{Q}_{55}\big|_2\left(H_2-\frac{4H_2^3}{3h^2}\right)\frac{d\phi}{dx} - b_{55}\big|_2 H_2\frac{d\phi}{dx} + d_2(x)$$

or

$$d_2(x) = d_1(x) + \left(H_2-\frac{4H_2^3}{3h^2}\right)\frac{d\phi}{dx}\left(\bar{Q}_{55}\big|_2-\bar{Q}_{55}\big|_1\right)$$

$$+ H_2\frac{d\phi}{dx}\left(b_{55}\big|_2-\cancelto{0}{b_{55}\big|_1}\right)$$

This points toward a cumulative effect, so that, for $k = 2 \rightarrow 2M+1$

$$d_k(x) = d_1(x) + \sum_{r=1}^{k-1}\Big\{H_{r+1}\left(b_{55}\big|_{r+1}-b_{55}\big|_r\right)$$

$$+\left(\bar{Q}_{55}\big|_{r+1}-\bar{Q}_{55}\big|_r\right)\left(H_{r+1}-\frac{4H_{r+1}^3}{3h^2}\right)\Big\}\frac{d\phi}{dx}$$

If we now define

$$\bar{\alpha}_1 = 0 \tag{2.3.13}$$

$$\bar{\alpha}_k = \sum_{r=1}^{k-1} \left\{ H_{r+1}\left(b_{55}\big|_{r+1} - b_{55}\big|_{r}\right) + \left(\bar{Q}_{55}\big|_{r+1} - \bar{Q}_{55}\big|_{r}\right)\left(H_{r+1} - \frac{4H_{r+1}^3}{3h^2}\right)\right\} \tag{2.3.14}$$

We get

$$d_k(x) = \left(\bar{\alpha}_k - \frac{h}{3}\bar{Q}_{55}\big|_1\right)\frac{d\phi}{dx} - \frac{h}{8}\frac{d\tau_L}{dx} + \frac{h}{8}\frac{d\tau_u}{dx} + \sigma_L \tag{2.3.15}$$

Utilizing (2.3.15) with (2.3.11) and setting $k = 2M+1$, $z=h/2$ for the upper surface, we have

$$\begin{aligned}\sigma_z\left(x,\tfrac{h}{2}\right)_{2M+1} = \sigma_u = & -\frac{d\tau_L}{dx}\left(-\frac{h}{8} - \frac{h^2}{8h} + \frac{h^3}{8h^2}\right) - \frac{d\tau_u}{dx}\left(-\frac{h}{8} + \frac{h^2}{8h} + \frac{h^3}{8h}\right) \\ & + \left[-\bar{Q}_{55}\big|_{2M+1}\left(\frac{h}{2} - \frac{4h^3}{3h^2}\right) - \cancelto{0}{b_{55}\big|_{2M+1}}\,\frac{h}{2}\right]\frac{d\phi}{dx} \\ & + \left\{\left(\bar{\alpha}_{2M+1} - \frac{h}{3}\bar{Q}_{55}\big|_1\right)\frac{d\phi}{dx} - \frac{h}{8}\frac{d\tau_L}{dx} + \frac{h}{8}\frac{d\tau_u}{dx} + \sigma_u\right\}\end{aligned}$$

or

$$\sigma_u(x) = \sigma_L + \left[\bar{\alpha}_{2M+1} - \frac{2h}{3}\bar{Q}_{55}\big|_1\right]\frac{d\phi}{dx} \tag{2.3.16}$$

See Section 3.1 for a further discussion of this equation. Equation (2.3.16) shows that only three of the four quantities σ_L, σ_u, τ_L, τ_u are independent; the fourth (σ_u in this derivation) is determined from the other three by the equilibrium equation (2.2.3) and the continuity condition (2.3.12).

With equations (2.3.1) and (2.3.8b), and (2.3.11)

and (2.3.15), we have expressions for τ_{xz} and σ_z in terms of known material properties, the surface shear stresses, and the function $\phi(x)$.

2.4 Calculation of In-Plane Displacement, $u(x,z)_k$ and Local Slope $\frac{\partial w}{\partial x}(x,z)|_k$

The functions u and $\frac{\partial w}{\partial x}$ will now be computed in terms of the known material properties, known temperature and moisture functions, the surface tractions, and the function ϕ. Using the plane-strain constitutive relation (2.1.17), we have

$$\varepsilon_{yz}|_k = 0 = \bar{S}_{44}\ \tau_{yz}|_k + \bar{S}_{45}\ \tau_{xz}|_k \tag{2.4.1a}$$

or

$$\tau_{yz}|_k = -\frac{\bar{S}_{45}}{\bar{S}_{44}}\Big|_k\ \tau_{xz}|_k \tag{2.4.1b}$$

We define
$$\Delta_k = -\frac{\bar{S}_{45}}{\bar{S}_{44}}\Big|_k \tag{2.4.2}$$

Thus, again using (2.1.17), we see that

$$\varepsilon_{xz}|_k = (\Delta+\bar{S}_{55})_k\ \tau_{xz}|_k \tag{2.4.3}$$

Using the definition (2.3.1) for $\tau_{xz}|_k$, and noting that $\bar{S}_{55}|_k \bar{Q}_{55}|_k = 1$, we have

$$\begin{aligned}\varepsilon_{xz}|_k &= (\Delta+\bar{S}_{55})_k \left[\tau_L\, f_1(z) + \tau_u\, f_2(z)\right] \\ &\quad + \left\{(1+\Delta\bar{Q}_{55})_k\, f_3(z) + (\Delta+\bar{S}_{55})_k\, b_{55}|_k\right\}\phi\end{aligned} \tag{2.4.4}$$

Next, we rearrange the strain-displacement equation (2.1.24),

$$\frac{\partial u}{\partial z}\Big|_k = -\frac{\partial w}{\partial x}\Big|_k + 2\varepsilon_{xz}\Big|_k \tag{2.4.5}$$

and substitute (2.4.4) into (2.4.5)

$$\begin{aligned}\frac{\partial u(x,z)_k}{\partial z} = {} & -\frac{\partial w}{\partial x}\Big|_k + (\Delta+\bar{S}_{55})_k\Big[\tau_L\Big(-\frac{1}{2}-\frac{2z}{h}+\frac{6z^2}{h^2}\Big) \\ & +\tau_u\Big(-\frac{1}{2}+\frac{2z}{h}+\frac{6z^2}{h^2}\Big)\Big] + 2(1+\Delta\bar{Q}_{55})_k\Big(1-\frac{4z^2}{h^2}\Big)\phi \\ & + 2(\Delta+\bar{S}_{55})_k\, b_{55}\big|_k\, \phi\end{aligned} \tag{2.4.6}$$

Integrating equation (2.4.6) we obtain

$$\begin{aligned}u(x,z)_k = {} & -\int^z \frac{\partial w}{\partial x}\Big|_k dz + (\Delta+\bar{S}_{55})_k\Big[\tau_L\Big(-\frac{z}{2}-\frac{z^2}{h}+\frac{2z^3}{h^2}\Big) \\ & +\tau_u\Big(-\frac{z}{2}+\frac{z^2}{h}+\frac{2z^3}{h^2}\Big)\Big] + 2(1+\Delta\bar{Q}_{55})_k\Big(z-\frac{4z^3}{3h^2}\Big)\phi \\ & + 2(\Delta+\bar{S}_{55})_k\, b_{55}\big|_k\, z\phi + \xi_k(x)\end{aligned} \tag{2.4.7}$$

Now continue to reduce the constitutive equation system by employing the other two plane-strain assumptions. From (2.1.17),

$$(\overset{0}{\cancel{\varepsilon_y}}-\alpha_y\Delta T-\beta_y m)_k = (\bar{S}_{12}\sigma_x + \bar{S}_{22}\sigma_y + \bar{S}_{23}\sigma_z + \bar{S}_{26}\tau_{xy})_k$$

or, for the k^{th} lamina

$$\sigma_y = -\frac{\bar{S}_{12}}{\bar{S}_{22}}\sigma_x - \frac{\bar{S}_{23}}{\bar{S}_{22}}\sigma_z - \frac{\bar{S}_{26}}{\bar{S}_{22}}\tau_{xy} - \frac{1}{\bar{S}_{22}}(\alpha_y \Delta T + \beta_y m) \tag{2.4.8}$$

Also, we have, for the k^{th} lamina,

$$\overset{0}{\cancel{\varepsilon_{xy}}} - \alpha_{xy}\Delta T - \beta_{xy} m = \bar{S}_{61}\sigma_x + \bar{S}_{62}\sigma_y + \bar{S}_{63}\sigma_z + \bar{S}_{66}\tau_{xy} \tag{2.4.9a}$$

or, substituting (2.4.8) for σ_y,

$$\begin{aligned} 0 = {} & \alpha_{xy}\Delta T + \beta_{xy} m + \bar{S}_{61}\sigma_x \\ & + \bar{S}_{62}\left[-\frac{\bar{S}_{12}}{\bar{S}_{22}}\sigma_x - \frac{\bar{S}_{23}}{\bar{S}_{22}}\sigma_z - \frac{\bar{S}_{26}}{\bar{S}_{22}}\tau_{xy} \right. \\ & \left. - \frac{1}{\bar{S}_{22}}(\alpha_y \Delta T + \beta_y m)\right] + \bar{S}_{63}\sigma_z + \bar{S}_{66}\tau_{xy} \end{aligned} \tag{2.4.9b}$$

or

$$\begin{aligned} \tau_{xy} = {} & \frac{\left(\bar{S}_{61} - \bar{S}_{62}\frac{\bar{S}_{12}}{\bar{S}_{22}}\right)}{\left(\bar{S}_{62}\frac{\bar{S}_{26}}{\bar{S}_{22}} - \bar{S}_{66}\right)}\sigma_x + \frac{\left(\bar{S}_{63} - \bar{S}_{62}\frac{\bar{S}_{23}}{\bar{S}_{22}}\right)}{\left(\bar{S}_{62}\frac{\bar{S}_{26}}{\bar{S}_{22}} - \bar{S}_{66}\right)}\sigma_z \\ & + \frac{(\alpha_{xy}\Delta T + \beta_{xy} m)}{\left(\bar{S}_{62}\frac{\bar{S}_{26}}{\bar{S}_{22}} - \bar{S}_{66}\right)} - \frac{\left(\frac{\bar{S}_{62}}{\bar{S}_{22}}\right)(\alpha_y \Delta T + \beta_y m)}{\left(\bar{S}_{62}\frac{\bar{S}_{26}}{\bar{S}_{22}} - \bar{S}_{66}\right)} \end{aligned} \tag{2.4.10}$$

Now proceed to the equation which will yield an expression for the local slope $\frac{\partial w}{\partial x}$. Using the equation for the normal strain:

$$\varepsilon_z - \alpha_z \Delta T - \beta_z m = \bar{S}_{13}\sigma_x + \bar{S}_{23}\sigma_y + \bar{S}_{33}\sigma_z + \bar{S}_{36}\tau_{xy} \tag{2.4.11a}$$

substitute first for σ_y, using (2.4.8),

$$\varepsilon_z - \alpha_z \Delta T - \beta_z m = \bar{S}_{13}\sigma_x + \bar{S}_{23}\Big[-\frac{\bar{S}_{12}}{\bar{S}_{22}}\sigma_x - \frac{\bar{S}_{23}}{\bar{S}_{22}}\sigma_z - \frac{\bar{S}_{26}}{\bar{S}_{22}}\tau_{xy} - \frac{1}{\bar{S}_{22}}(\alpha_y \Delta T + \beta_y m)\Big] + \bar{S}_{33}\sigma_z + \bar{S}_{36}\tau_{xy}$$

or

$$\varepsilon_z - \alpha_z \Delta T - \beta_z m = \Big(\bar{S}_{13} - \bar{S}_{23}\frac{\bar{S}_{12}}{\bar{S}_{22}}\Big)\sigma_x + \Big(\bar{S}_{33} - \bar{S}_{23}\frac{\bar{S}_{23}}{\bar{S}_{22}}\Big)\sigma_z + \Big(\bar{S}_{36} - \bar{S}_{23}\frac{\bar{S}_{26}}{\bar{S}_{22}}\Big)\tau_{xy} - \frac{\bar{S}_{23}}{\bar{S}_{22}}(\alpha_y \Delta T + \beta_y m) \qquad (2.4.11b)$$

where the quantities in parentheses are often referred to as the "reduced" flexibility terms.

Now substitute (2.4.10) for τ_{xy} into (2.4.11b)

$$\varepsilon_z - \alpha_z \Delta T - \beta_z m = \Big(\bar{S}_{13} - \bar{S}_{23}\frac{\bar{S}_{12}}{\bar{S}_{22}}\Big)\sigma_x + \Big(\bar{S}_{33} - \frac{\bar{S}_{23}^2}{\bar{S}_{22}}\Big)\sigma_z + \Big(\bar{S}_{36} - \bar{S}_{23}\frac{\bar{S}_{26}}{\bar{S}_{22}}\Big)\Bigg[\frac{\Big(\bar{S}_{61} - \bar{S}_{62}\frac{\bar{S}_{12}}{\bar{S}_{22}}\Big)\sigma_x}{\Big(\bar{S}_{62}\frac{\bar{S}_{26}}{\bar{S}_{22}} - \bar{S}_{66}\Big)} + \frac{\Big(\bar{S}_{63} - \bar{S}_{62}\frac{\bar{S}_{23}}{\bar{S}_{22}}\Big)}{\Big(\bar{S}_{62}\frac{\bar{S}_{26}}{\bar{S}_{22}} - \bar{S}_{66}\Big)}\sigma_z + \frac{(\alpha_{xy}\Delta T + \beta_{xy} m)}{\Big(\bar{S}_{62}\frac{\bar{S}_{26}}{\bar{S}_{22}} - \bar{S}_{66}\Big)} - \frac{\Big(\frac{\bar{S}_{62}}{\bar{S}_{22}}\Big)(\alpha_y \Delta T + \beta_y m)}{\Big(\bar{S}_{62}\frac{\bar{S}_{26}}{\bar{S}_{22}} - \bar{S}_{66}\Big)}\Bigg] - \Big(\frac{\bar{S}_{23}}{\bar{S}_{22}}\Big)(\alpha_y \Delta T + \beta_y m)$$

or, introducing a compact notation, for the k^{th} lamina,

$$\varepsilon_z\Big|_k = (\bar{S}'_{13}\sigma_x + \bar{S}'_{33}\sigma_z + \alpha'_z\Delta T + \beta'_z m)_k \qquad (2.4.11c)$$

where, including a tracing constant "a" to follow off-axis effects,

$$\bar{S}'_{13} = \bar{S}_{13} - a\left[\frac{\bar{S}_{23}\bar{S}_{12}}{\bar{S}_{22}} + \frac{(\bar{S}_{36}-\bar{S}_{23}\,\frac{\bar{S}_{66}}{\bar{S}_{22}})}{(\bar{S}_{66}-\bar{S}_{62}\,\frac{\bar{S}_{26}}{\bar{S}_{22}})}\,(\bar{S}_{61}-\bar{S}_{62}\,\frac{\bar{S}_{12}}{\bar{S}_{22}})\right] \qquad (2.4.12a)$$

$$\bar{S}'_{33} = \bar{S}_{33} - a\left[\frac{\bar{S}^2_{23}}{\bar{S}_{22}} + \frac{(\bar{S}_{36}-\bar{S}_{23}\,\frac{\bar{S}_{26}}{\bar{S}_{22}})}{(\bar{S}_{66}-\bar{S}_{62}\,\frac{\bar{S}_{26}}{\bar{S}_{22}})}\,(\bar{S}_{63}-\bar{S}_{62}\,\frac{\bar{S}_{23}}{\bar{S}_{22}})\right] \qquad (2.4.12b)$$

$$\alpha'_z = \alpha_z + \frac{(\bar{S}_{36}-\bar{S}_{23}\,\frac{\bar{S}_{26}}{\bar{S}_{22}})}{(\bar{S}_{62}\,\frac{\bar{S}_{26}}{\bar{S}_{22}}-\bar{S}_{66})}\,\alpha_{xy}$$

$$- \frac{(\frac{\bar{S}_{62}}{\bar{S}_{22}})(\bar{S}_{36}-\bar{S}_{23}\,\frac{\bar{S}_{26}}{\bar{S}_{22}})}{(\bar{S}_{62}\,\frac{\bar{S}_{26}}{\bar{S}_{22}} - \bar{S}_{66})}\,\alpha_y - \frac{\bar{S}_{23}}{\bar{S}_{22}}\,\alpha_y \qquad (2.4.13a)$$

$$\beta_z' = \beta_z + \frac{(\bar{S}_{36}-\bar{S}_{23}\frac{\bar{S}_{26}}{\bar{S}_{22}})}{(\bar{S}_{62}\frac{\bar{S}_{26}}{\bar{S}_{22}} - \bar{S}_{66})}\beta_{xy}$$

$$- \frac{(\frac{\bar{S}_{62}}{\bar{S}_{22}})(\bar{S}_{36}-\bar{S}_{23}\frac{\bar{S}_{26}}{\bar{S}_{22}})}{(\bar{S}_{62}\frac{\bar{S}_{26}}{\bar{S}_{22}} - \bar{S}_{66})}\beta_y - \frac{\bar{S}_{23}}{\bar{S}_{22}}\beta_y \qquad (2.4.13b)$$

If we had used a plane-stress analysis ($\sigma_z = \tau_{xz} = \tau_{yz} = 0$) with the Kirchoff linear strain hypothesis, this would have lead to a laminate effective α, "α^L", and effective β, "β^L", and thus we would have obtained

$$\varepsilon_y = \alpha_y^L \Delta T + \beta_y^L m$$

$$\varepsilon_{xy} = \alpha_{xy}^L \Delta T + \beta_{xy}^L m$$

$$\varepsilon_{yz} = 0$$

The difference between the two methods may be easily accomodated in equations (2.4.8) and following by substituting $(\alpha_y^L - \alpha_y|_k)$ for $-\alpha_y|_k$, $(\beta_y^L - \beta_y|_k)$ for $-\beta_y|_k$, $(\alpha_{xy}^L - \alpha_{xy}|_k)$ for $-\alpha_{xy}|_k$, and $(\beta_{xy}^L - \beta_{xy}|_k)$ for $-\beta_{xy}|_k$. For any symmetric laminate, $\alpha_{xy}^L = \beta_{xy}^L = 0$

Continuing to use the plane-strain analysis to solve for $\frac{\partial w}{\partial x}$, we differentiate (2.4.11) with respect to x, and use (2.4.12) and (2.4.13):

$$\frac{\partial}{\partial x}\,\varepsilon_z\Big|_k^{o} = \left[\bar{S}'_{13}\frac{\partial \sigma_x}{\partial x} + \bar{S}'_{33}\frac{\partial \sigma_z}{\partial x} + \alpha'_z\frac{\partial \Delta T}{\partial x} + \beta'_z\frac{\partial m}{\partial x}\right]_k \tag{2.4.14}$$

Now differentiate equation (2.3.10) with respect to x;

$$\frac{\partial}{\partial x}\,\sigma_z\Big|_k = -\frac{\partial^2}{\partial x^2}\int^{z}\tau_{xz}\Big|_k\,dz + \frac{d}{dx}\,d_k(x) \tag{2.4.15}$$

and also use the equilibrium equation

$$\frac{\partial\sigma_x}{\partial x}\Big|_k = -\frac{\partial\tau_{xz}}{\partial z}\Big|_k \tag{2.2.1}$$

in an effort to get (2.4.14) in terms of the unknown functions already defined. Substituting (2.2.1) and (2.4.15) into (2.4.14) and using the definition (2.3.1) for $\tau_{xz}|_k$,

$$\begin{aligned}
\frac{\partial}{\partial x}\,\varepsilon_z\Big|_k = \Big[\, & \bar{S}'_{13}\Big|_k\left(\frac{1}{h}-\frac{6z}{h^2}\right)\tau_L + \bar{S}'_{13}\Big|_k\left(-\frac{1}{h}-\frac{6z}{h^2}\right)\tau_u \\
& + \frac{8\,\bar{S}'_{13}\,\bar{Q}_{55}|_k}{h^2}\,z\,\phi\Big] + \Big[-\bar{S}'_{33}\Big|_k\left(-\frac{z}{4}-\frac{z^2}{2h}+\frac{z^3}{h^2}\right)\frac{d^2\tau_L}{dx^2} \\
& - \bar{S}'_{33}\Big|_k\left(-\frac{z}{4}+\frac{z^2}{2h}+\frac{z^3}{h^2}\right)\frac{d^2\tau_u}{dx^2} - \bar{S}'_{33}\,\bar{Q}_{55}\Big|_k\left(z-\frac{4z^3}{3h^2}\right)\frac{d^2\phi}{dx^2} \\
& - \bar{S}'_{33}\Big|_k\,b_{55}\Big|_k\,z\,\frac{d^2\phi}{dx^2} + \bar{S}'_{33}\Big|_k\left(\bar{\alpha}_k-\frac{h}{3}\bar{Q}_{55}\Big|_1\right)\frac{d^2\phi}{dx^2} \\
& + \bar{S}'_{33}\Big|_k\,\frac{d\sigma_L}{dx} - \bar{S}'_{33}\Big|_k\,\frac{h}{8}\,\frac{d^2\tau_L}{dx^2} + \bar{S}'_{33}\Big|_k\,\frac{h}{8}\,\frac{d^2\tau_u}{dx^2}\Big] \\
& + \alpha'_z\Big|_k\,\frac{\partial\Delta T}{\partial x} + \beta'_z\Big|_k\,\frac{\partial m}{\partial x}
\end{aligned} \tag{2.4.16}$$

By use of the strain-displacement relation (2.1.22) we have

$$\frac{\partial \varepsilon_z}{\partial x}\Big|_k = \frac{\partial^2 w}{\partial x \partial z}\Big|_k \qquad (2.4.17)$$

Given the continuity of derivatives of "w" within the lamina, we may use (2.4.17) with (2.4.16) and integrate with respect to z ,

$$\begin{aligned}
\frac{\partial}{\partial x} w(x,z)_k = {} & \bar{S}'_{13}\big|_k \left(\frac{z}{h} - \frac{3z^2}{h^2}\right)\tau_L + \bar{S}'_{13}\big|_k \left(\frac{z}{h} - \frac{3z^2}{h^2}\right)\tau_u \\
& + \bar{S}'_{33}\big|_k \left(\frac{z^2}{8} + \frac{z^3}{6h} - \frac{z^4}{4h^2} - \frac{zh}{8}\right)\frac{d^2\tau_L}{dx^2} \\
& + \bar{S}'_{33}\big|_k \left(\frac{z^2}{8} - \frac{z^3}{6h} - \frac{z^4}{4h^2} + \frac{zh}{8}\right)\frac{d^2\tau_u}{dx^2} \\
& + \frac{4\,\bar{S}'_{13}\,\bar{Q}_{55}\big|_k}{h^2} z^2 \phi - \bar{S}'_{13}\big|_k \Big[\bar{Q}_{55}\big|_k \left(\frac{z^2}{2} - \frac{z^4}{3h^2}\right) \\
& - \alpha_k z + \bar{Q}_{55}\big|_1 \frac{hz}{3} + b_{55}\big|_k \frac{z^2}{2}\Big]\frac{d^2\phi}{dx^2} \\
& + \bar{S}'_{33}\big|_k z \frac{d\sigma_L}{dx} + \alpha'_z\big|_k \int^z \frac{\partial \Delta T}{\partial x} dz \\
& + \beta'_z\big|_k \int^z \frac{\partial m}{\partial x} dz + \bar{\psi}_k(x)
\end{aligned} \qquad (2.4.18)$$

where $\bar{\psi}_k$ is a function of integration to be found.

The functions ΔT and m are assumed smooth throughout the laminate and assumed to be polynomial in z , so that the indefinite integrals with respect to z may be written as

$$\int^z \frac{\partial}{\partial x}\Delta T\, dz = \int_0^z \frac{\partial}{\partial x}\Delta T\, dz + \chi_1(x) \qquad (2.4.19)$$

$$\int^{z} \frac{\partial}{\partial x} m \, dz = \int_{0}^{z} \frac{\partial}{\partial x} m \, dz + \chi_{2}(x) \tag{2.4.20}$$

where the integral evaluated at the lower limit, $z = 0$, will be zero due to the polynomial in z in the integral.

Now examine the midsurface, $z = 0$, $k = m + 1$:

$$\frac{\partial}{\partial x} w(x,0)_{M+1} \equiv \frac{dw^{\circ}}{dx} = \bar{\psi}_{M+1} + \chi_{1} + \chi_{2}$$

If we redefine the functions of integration to include the functions χ_1 *and* χ_2, we simplify the form, since we do not need to solve for χ_1 or χ_2. Thus, set

$$\psi_{k} = \bar{\psi}_{k} + \chi_{1} + \chi_{2} \tag{2.4.21}$$

for $k = 1 \rightarrow 2M+1$

Thus we have

$$\psi_{M+1}(x) = \frac{dw^{\circ}}{dx} \tag{2.4.22}$$

For convenience, we define the following functions

$$\int_{0}^{z} \frac{\partial}{\partial x} \Delta T \, dz = \bar{g}(x,z) \tag{2.4.23}$$

$$\int_0^z \frac{\partial}{\partial x} m \, dz = \bar{h}(x,z) \qquad (2.4.24)$$

where $\bar{g}$ and $\bar{h}$ are continuous everywhere in the laminate.

To solve for the $\psi_k(x)$ in the other laminae, we must make an assumption. The quantity $\frac{\partial w}{\partial x}$ represents a local slope which is assumed to be continuous at any interface within the laminate. If the slope is not continuous, this implies delamination and void formation. To build up the solution for the ψ_k, proceed to the M+1, M+2 interface, $z = H_{M+2}$ and invoke continuity of $\frac{\partial w}{\partial x}$ using (2.4.18) and (2.4.22):

$$\frac{\partial w}{\partial x}(x, H_{M+2})_{M+1} - \frac{\partial w}{\partial x}(x, H_{M+2})_{M+2} = 0$$

or

$$\begin{aligned}
0 = {} & \psi_{M+1} - \psi_{M+2} + \left(\bar{S}'_{13}\big|_{M+1} - \bar{S}'_{13}\big|_{M+2}\right)\left(\frac{H_{M+2}}{h} - \frac{3H_{M+2}^2}{h^2}\right)\tau_L \\
& + \left(\bar{S}'_{13}\big|_{M+1} - \bar{S}'_{13}\big|_{M+2}\right)\left(-\frac{H_{M+2}}{h} - \frac{3H_{M+2}^2}{h^2}\right)\tau_u \\
& + \left(\bar{S}'_{33}\big|_{M+1} - \bar{S}'_{33}\big|_{M+2}\right)\left(\frac{H_{M+2}^2}{8} + \frac{H_{M+2}^3}{6h} - \frac{H_{M+2}^4}{4h^2} - \frac{H_{M+2}h}{8}\right)\frac{d^2\tau_L}{dx^2} \\
& + \left(\bar{S}'_{33}\big|_{M+1} - \bar{S}'_{33}\big|_{M+2}\right)\left(\frac{H_{M+2}^2}{8} - \frac{H_{M+2}^3}{6h} - \frac{H_{M+2}^4}{4h^2} + \frac{H_{M+2}h}{8}\right)\frac{d^2\tau_u}{dx^2} \\
& + \frac{4}{h^2} H_{M+2}^2 \left(\bar{S}'_{13}\bar{Q}_{55}\big|_{M+1} - \bar{S}'_{13}\bar{Q}_{55}\big|_{M+2}\right)\phi \\
& - \Big[\left(\bar{S}'_{33}\bar{Q}_{55}\big|_{M+1} - \bar{S}'_{33}\bar{Q}_{55}\big|_{M+2}\right)\left(\frac{H_{M+2}^2}{2} - \frac{H_{M+2}^4}{3h^2}\right) \\
& - \left(\bar{S}'_{33}\bar{\alpha}\big|_{M+1} - \bar{S}'_{33}\bar{\alpha}\big|_{M+2}\right)H_{M+2}
\end{aligned}$$

(over)

$$+ \bar{Q}_{55}\big|_1 \left(\bar{S}'_{33}\big|_{M+1} - \bar{S}'_{33}\big|_{M+2}\right) \frac{h H_{M+2}}{3}$$

$$\left. + \left(\bar{S}'_{33}\, b_{55}\big|_{M+1} - \bar{S}'_{33}\, b_{55}\big|_{M+2}\right) \frac{H_{M+2}^2}{2} \right] \frac{d^2\phi}{dx^2}$$

$$+ \left(\bar{S}'_{33}\big|_{M+1} - \bar{S}'_{33}\big|_{M+2}\right) H_{M+2} \frac{d\sigma_L}{dx} + \left(\alpha'_z\big|_{M+1} - \alpha'_z\big|_{M+2}\right) \bar{g}(x, H_{M+2})$$

$$+ \left(\beta'_z\big|_{M+1} - \beta'_z\big|_{M+2}\right) \bar{h}(x, H_{M+2})$$

Thus, we may see a general pattern for the cumulative effect layer by layer. Introduce a tracing constant "b" to determine the effect of transverse normal strain, ε_z. We will use the form:

$$\psi_k(x) = \frac{dw^\circ}{dx} + \rho_k \frac{d^2\phi}{dx^2} + \mu_k \phi + \eta_k \frac{d^2\tau_L}{dx^2} + \bar{\eta}_k \frac{d^2\tau_u}{dx^2} + \delta_k \tau_L + \bar{\delta}_k \tau_u + \varepsilon_k \frac{d\sigma_L}{dx} + g_k(x) \quad (2.4.25)$$

where, for k=M+2→2M+1 ,

$$\rho_k = b \sum_{r=M+1}^{k-1} \Big\{ \left(\bar{S}'_{33}\, \bar{Q}_{55}\big|_{r+1} - \bar{S}'_{33}\, \bar{Q}_{55}\big|_r\right)\left(\frac{H_{r+1}^2}{2} - \frac{H_{r+1}^4}{3h^2}\right) + \left(\bar{S}'_{33}\bar{\alpha}\big|_r - \bar{S}'_{33}\bar{\alpha}\big|_{r+1}\right) H_{r+1} + \bar{Q}_{55}\big|_1 \left(\bar{S}'_{33}\big|_{r+1} - \bar{S}'_{33}\big|_r\right)\frac{h H_{r+1}}{3} + \left(\bar{S}'_{33}\, b_{55}\big|_{r+1} - \bar{S}'_{33}\, b_{55}\big|_r\right)\frac{H_{r+1}^2}{2} \Big\} \quad (2.4.26)$$

$$\mu_k = b\sum_{r=M+1}^{k-1} \left\{ \frac{4}{h^2} H_{r+1}^2 \left(\bar{S}_{13}' \bar{Q}_{55}\big|_r - \bar{S}_{13}' \bar{Q}_{55}\big|_{r+1} \right) \right\} \qquad (2.4.27)$$

$$\eta_k = b\sum_{r=M+1}^{k-1} \left\{ \left(\bar{S}_{33}'\big|_r - \bar{S}_{33}'\big|_{r+1} \right) \left(\frac{H_{r+1}^2}{8} + \frac{H_{r+1}^3}{6h} - \frac{H_{r+1}^4}{4h^2} - \frac{hH_{r+1}}{8} \right) \right\} \qquad (2.4.28)$$

$$\bar{\eta}_k = b\sum_{r=M+1}^{k-1} \left\{ \left(\bar{S}_{33}'\big|_r - \bar{S}_{33}'\big|_{r+1} \right) \left(\frac{H_{r+1}^2}{8} - \frac{H_{r+1}^3}{6h} - \frac{H_{r+1}^4}{4h^2} + \frac{hH_{r+1}}{8} \right) \right\} \qquad (2.4.29)$$

$$\delta_k = b\sum_{r=M+1}^{k-1} \left\{ \left(\bar{S}_{13}'\big|_r - \bar{S}_{13}'\big|_{r+1} \right) \left(\frac{H_{r+1}}{h} - \frac{3H_{r+1}^2}{h^2} \right) \right\} \qquad (2.4.30)$$

$$\bar{\delta}_k = b\sum_{r=M+1}^{k-1} \left\{ \left(\bar{S}_{13}'\big|_r - \bar{S}_{13}\big|_{r+1} \right) \left(-\frac{H_{r+1}}{h} - \frac{3H_{r+1}^2}{h^2} \right) \right\} \qquad (2.4.31)$$

$$\varepsilon_k = b\sum_{r=M+1}^{k-1} \left\{ \left(\bar{S}_{33}'\big|_r - \bar{S}_{33}'\big|_{r+1} \right) H_{r+1} \right\} \qquad (2.4.32)$$

$$g_k(x) = b\sum_{r=M+1}^{k-1} \left\{ \left(\alpha_z'\big|_r - \alpha_z'\big|_{r+1} \right) \bar{g}(x, H_{r+1}) + \left(\beta_z'\big|_r - \beta_z'\big|_{r+1} \right) \bar{h}(x, H_{r+1}) \right\} \qquad (2.4.33)$$

Now examine the M+1, M interface, $z=H_{M+1}$, to set up similar constants for $k = M \to 1$ (lower laminae); as stated earlier, the slopes must be equal

$$\frac{\partial w}{\partial x}(x, H_{M+1})_{M+1} - \frac{\partial w}{\partial x}(x, H_{M+1})_{M} = 0$$

or,

$$
\begin{aligned}
0 = {} & \psi_{M+1} - \psi_M + \left(\bar{S}'_{13}\big|_{M+1} - \bar{S}'_{13}\big|_M \right) \left(\frac{H_{M+1}}{h} - \frac{3H_{M+1}^2}{h^2} \right) \tau_L \\
& + \left(\bar{S}'_{13}\big|_{M+1} - \bar{S}'_{13}\big|_M \right) \left(-\frac{H_{M+1}}{h} - \frac{3H_{M+1}^2}{h^2} \right) \tau_u \\
& + \left(\bar{S}'_{33}\big|_{M+1} - \bar{S}'_{33}\big|_M \right) \left(\frac{H_{M+1}^2}{8} + \frac{H_{M+1}^3}{6h} - \frac{H_{M+1}^4}{4h^2} - \frac{H_{M+1}\,h}{8} \right) \frac{d^2 \tau_L}{dx^2} \\
& + \left(\bar{S}'_{33}\big|_{M+1} - \bar{S}'_{33}\big|_M \right) \left(\frac{H_{M+1}^2}{8} - \frac{H_{M+1}^3}{6h} - \frac{H_{M+1}^4}{4h^2} + \frac{H_{M+1}\,h}{8} \right) \frac{d^2 \tau_u}{dx^2} \\
& + \frac{4}{h^2} H_{M+1}^2 \left(\bar{S}'_{13} \bar{Q}_{55}\big|_{M+1} - \bar{S}'_{13} \bar{Q}_{55}\big|_M \right) \phi \\
& - \Big[\left(\bar{S}'_{33} \bar{Q}_{55}\big|_{M+1} - \bar{S}'_{33} \bar{Q}_{55}\big|_M \right) \left(\frac{H_{M+1}^2}{2} - \frac{H_{M+1}^4}{3h^2} \right) \\
& \quad - \left(\bar{S}'_{33} \bar{\alpha}\big|_{M+1} - \bar{S}'_{33} \bar{\alpha}\big|_M \right) H_{M+1} + \bar{Q}_{55}\big|_1 \left(\bar{S}'_{33}\big|_{M+1} - \bar{S}'_{33}\big|_M \right) \frac{h\, H_{M+1}}{3} \\
& \quad + \left(\bar{S}'_{33} b_{55}\big|_{M+1} - \bar{S}'_{33} b_{55}\big|_M \right) \frac{H_{M+1}^2}{2} \Big] \frac{d^2 \phi}{dx^2} \\
& + \left(\bar{S}'_{33}\big|_{M+1} - \bar{S}'_{33}\big|_M \right) H_{M+1} \frac{d \sigma_L}{dx} + \left(\alpha'_z\big|_{M+1} - \alpha'_z\big|_M \right) \bar{g}(x, H_{M+1}) \\
& + \left(\beta'_z\big|_{M+1} - \beta'_z\big|_M \right) \bar{h}(x, H_{M+1})
\end{aligned}
$$

This suggests the cumulative effect for the lower laminae. Defining quantities as in equation (2.4.25) for k = M → 1 ,

$$\psi_k(x) = \frac{dw^\circ}{dx} + \rho_k \frac{d^2\phi}{dx^2} + \mu_k \phi + \eta_k \frac{d^2\tau_L}{dx^2} + \bar{\eta}_k \frac{d^2\tau_u}{dx^2} + \delta_k \tau_L + \bar{\delta}_k \tau_u + \epsilon_k \frac{d\sigma_L}{dx} + g_k(x) \quad (2.4.25)$$

where we again use the tracing constant "b" , to determine the effect of transverse normal strain. We define the terms:

$$\rho_k = b\sum_{r=M+1}^{K+1} \Big\{ (\bar{S}'_{33}\bar{Q}_{55}|_{r-1} - \bar{S}'_{33}\bar{Q}_{55}|_r)\left(\frac{H_r^2}{2} - \frac{H_r^4}{3h^2}\right) + (\bar{S}'_{33}\bar{\alpha}|_r - \bar{S}'_{33}\bar{\alpha}|_{r-1}) H_r + \bar{Q}_{55}|_1 (\bar{S}'_{33}|_{r-1} - \bar{S}'_{33}|_r)\frac{hH_r}{3} + (\bar{S}'_{33} b_{55}|_{r-1} - \bar{S}'_{33} b_{55}|_r)\frac{H_r^2}{2} \Big\} \quad (2.4.34)$$

$$\mu_k = b\sum_{r=M+1}^{K+1} \left\{ \frac{4H_r^2}{h^2} (\bar{S}'_{13}\bar{Q}_{55}|_r - \bar{S}'_{13}\bar{Q}_{55}|_{r-1}) \right\} \quad (2.4.35)$$

$$\eta_k = b\sum_{r=M+1}^{K+1} \left\{ (\bar{S}'_{33}|_r - \bar{S}'_{33}|_{r-1})\left(\frac{H_r^2}{8} + \frac{H_r^3}{6h} - \frac{H_r^4}{4h^2} - \frac{hH_r}{8}\right) \right\} \quad (2.4.36)$$

$$\bar{\eta}_k = b\sum_{r=M+1}^{K+1} \left\{ (\bar{S}'_{33}|_r - \bar{S}'_{33}|_{r-1})\left(\frac{H_r^2}{8} - \frac{H_r^3}{6h} - \frac{H_r^4}{4h^2} + \frac{hH_r}{8}\right) \right\} \quad (2.4.37)$$

$$\delta_k = b\sum_{r=M+1}^{K+1} \left\{ (\bar{S}'_{13}|_r - \bar{S}'_{13}|_{r-1})\left(\frac{H_r}{h} - \frac{3H_r^2}{h^2}\right) \right\} \quad (2.4.38)$$

$$\bar{\delta}_k = b \sum_{r=M+1}^{k+1} \left\{ \left(\bar{S}'_{13}\big|_r - \bar{S}'_{13}\big|_{r-1} \right) \left(-\frac{H_r}{h} - \frac{3H_r^2}{h^2} \right) \right\} \tag{2.4.39}$$

$$\varepsilon_k = b \sum_{r=M+1}^{k+1} \left\{ \left(\bar{S}'_{33}\big|_r - \bar{S}'_{33}\big|_{r-1} \right) H_r \right\} \tag{2.4.40}$$

$$g_k(x) = b \sum_{r=M+1}^{k+1} \left\{ \left(\alpha'_z\big|_r - \alpha'_z\big|_{r-1} \right) \bar{g}(x, H_r) + \left(\beta'_z\big|_r - \beta'_z\big|_{r-1} \right) \bar{h}(x, H_r) \right\} \tag{2.4.41}$$

Note that μ_k is the only constant symmetric about the midsurface, i.e. $\mu_{2M+2-k} = \mu_k$. The ε_k are anti-symmetric, i.e. $\varepsilon_{2M+2-k} = -\varepsilon_k$. All the others - ρ_k , η_k , $\bar{\eta}_k$, δ_k and $\bar{\delta}_k$ are neither symmetric nor anti-symmetric.

Thus, we have implicit in our definition of ψ_k a tracing constant "b" which, if set to zero, will force all $\psi_k = \frac{dw^\circ}{dx}$. If we re-write (2.4.18) to employ the definitions (2.4.25) thru (2.4.41),

$$
\begin{aligned}
\frac{\partial w(x,z)_k}{\partial x} = {} & \frac{dw^{\circ}}{dx} + \left[\bar{S}'_{13}\Big|_k \left(\frac{z}{h} - \frac{3z^2}{h^2} \right) + \delta_k \right] \tau_L \\
& + \left[\bar{S}'_{13}\Big|_k \left(-\frac{z}{h} - \frac{3z^2}{h^2} \right) + \bar{\delta}_k \right] \tau_u \\
& + \left[\bar{S}'_{33}\Big|_k \left(\frac{z^2}{8} + \frac{z^3}{6h} - \frac{z^4}{4h^2} - \frac{zh}{8} \right) + \eta_k \right] \frac{d^2\tau_L}{dx^2} \\
& + \left[\bar{S}'_{33}\Big|_k \left(\frac{z^2}{8} - \frac{z^3}{6h} - \frac{z^4}{4h^2} + \frac{zh}{8} \right) + \bar{\eta}_k \right] \frac{d^2\tau_u}{dx^2} \\
& + \left[\frac{4\bar{S}'_{13}\bar{Q}_{55}\big|_k}{h^2} z^2 + \mu_k \right] \phi \\
& + \Bigg\{ -\bar{S}'_{33}\Big|_k \Bigg[\bar{Q}_{55}\Big|_k \left(\frac{z^2}{2} - \frac{z^4}{3h^2} \right) - \bar{\alpha}_k z \\
& \qquad + \bar{Q}_{55}\Big|_1 \frac{hz}{3} + b_{55}\Big|_k \frac{z^2}{2} \Bigg] + \rho_k \Bigg\} \frac{d^2\phi}{dx^2} \\
& + \left[\bar{S}'_{33}\Big|_k z + \varepsilon_k \right] \frac{d\sigma_L}{dx} + g_k(x) \\
& + \alpha'_z\Big|_k \bar{g}(x,z) + \beta'_z\Big|_k \bar{h}(x,z)
\end{aligned}
\qquad (2.4.42)
$$

where the function g_k contains both temperature and moisture functions of integration within it.

Now integrate equation (2.4.22) to supply the last item needed to find $u(x,z)$ in equation (2.4.7). This is done to obtain an equation for u in terms of the known material properties, the functions ϕ, τ_u τ_L, and σ_L, and the known functions m and ΔT.

$$
\begin{aligned}
\int^{z} \frac{\partial w(x,z)_k}{\partial x}\, dz = {} & z \frac{dw^{\circ}}{dx} + \left[\bar{S}'_{13}\big|_k \left(\frac{z^2}{2h} - \frac{z^3}{h^2} \right) + \delta_k z \right] \tau_L \\
& + \left[\bar{S}'_{13}\big|_k \left(\frac{-z^2}{2h} - \frac{z^3}{h^2} \right) + \bar{\delta}_k z \right] \tau_u \\
& + \left[\bar{S}'_{33}\big|_k \left(\frac{z^3}{24} + \frac{z^4}{24h} - \frac{z^5}{20h^2} - \frac{z^2 h}{16} \right) + \eta_k z \right] \frac{d^2 \tau_L}{dx^2} \\
& + \left[\bar{S}'_{33}\big|_k \left(\frac{z^3}{24} - \frac{z^4}{24h} - \frac{z^5}{20h^2} + \frac{z^2 h}{16} \right) + \bar{\eta}_k z \right] \frac{d^2 \tau_u}{dx^2} \\
& + \left[\frac{4 \bar{S}'_{13} \bar{Q}_{55}\big|_k}{3h^2} z^3 + \mu_k z \right] \phi \\
& + \Big\{ - \bar{S}'_{33}\big|_k \Big[\bar{Q}_{55}\big|_k \left(\frac{z^3}{6} - \frac{z^5}{15h^2} \right) - \bar{\alpha}_k \frac{z^2}{2} \\
& \qquad + \bar{Q}_{55}\big|_1 \frac{h z^2}{6} + b_{55}\big|_k \frac{z^3}{6} + \rho_k z \Big\} \frac{d^2 \phi}{dx^2} \\
& + \left[\bar{S}'_{33}\big|_k \frac{z^2}{2} + \varepsilon_k z \right] \frac{d\sigma_L}{dx} + z\, g_k(x) \\
& + \alpha'_z\big|_k \int^{z} \bar{g}(x,z)\, dz + \beta'_z\big|_k \int^{z} \bar{h}(x,z)\, dz + \xi_k(x)
\end{aligned}
$$

(2.4.43)

In an effort to simplify the function of integration, refer to the integrals below, noting the continuity of the integrand;

$$\int^{z} \bar{g}(x,z)\, dz = \int_0^{z} \bar{g}(x,z)\, dz + \mathcal{X}_3(x) \qquad (2.4.44)$$

$$\int^{z} \bar{h}(x,z)\, dz = \int_0^{z} \bar{h}(x,z)\, dz + \mathcal{X}_4(x) \qquad (2.4.45)$$

where, due to the assumed polynomial form in z of $\bar{g}_k$ and $\bar{h}_k$, the definite integral evaluated at the lower limit $z=0$ is zero. Now combine the functions of integration by letting:

$$C_k(x) = \bar{\xi}_k(x) - \left(\xi_k(x) + \mathcal{X}_3(x) + \mathcal{X}_4(x)\right) \tag{2.4.46}$$

where the $C_k(x)$ are unknown functions.

We now have an expression for $u(x,z)_k$ for a plane-strain plate element under the combined effects of transverse, normal, axial, temperature and thermal loadings. Using (2.4.7) (2.4.43), and (2.4.46), we obtain the expression

$$\begin{aligned} u(x,z)_k = {} & C_k(x) - z\frac{dw^\circ}{dx} + \Big[\bar{S}'_{13}\big|_k \left(-\frac{z^2}{2h} + \frac{z^3}{h^2}\right) \\ & + \left(\Delta + \bar{S}_{55}\right)_k \left(-\frac{z}{2} - \frac{z^2}{h} + \frac{2z^3}{h^2}\right) - \delta_k z \Big] \tau_L \\ & + \Big[\bar{S}'_{13}\big|_k \left(\frac{z^2}{h} + \frac{z^3}{h^2}\right) + \left(\Delta + \bar{S}_{55}\right)_k \left(-\frac{z}{2} + \frac{z^2}{h} + \frac{2z^3}{h^2}\right) - \bar{\delta}_k z \Big] \tau_u \\ & + \Big[\bar{S}'_{33}\big|_k \left(-\frac{z^3}{24} - \frac{z^4}{24h} + \frac{z^5}{20h^2} + \frac{hz^2}{16}\right) - \eta_k z \Big] \frac{d^2\tau_L}{dx^2} \\ & + \Big[\bar{S}'_{33}\big|_k \left(-\frac{z^3}{24} + \frac{z^4}{24h} + \frac{z^5}{20h^2} - \frac{hz^2}{16}\right) - \bar{\eta}_k z \Big] \frac{d^2\tau_u}{dx^2} \\ & + \Big[\left(2 + 2\Delta\bar{Q}_{55}\right)_k \left(z - \frac{4z^3}{3h^2}\right) + \left(2\Delta + 2\bar{S}_{55}\right)_k b_{55}\big|_k z \\ & - \frac{4\bar{S}'_{13}\bar{Q}_{55}\big|_k}{3h^2} z^3 - \mu_k z \Big] \phi \end{aligned}$$

(over)

$$+\Big\{ \bar{S}'_{33}\big|_k \Big[\bar{Q}_{55}\big|_k \Big(\frac{z^3}{6} - \frac{z^5}{15h^2} \Big) - \frac{\bar{\alpha}_k z^2}{2}$$

$$+ \bar{Q}_{55}\big|_1 \frac{h z^2}{6} + b_{55}\big|_k \frac{z^3}{6} - \rho_k z \Big\} \frac{d^2\phi}{dx^2}$$

$$+ \Big[-\bar{S}'_{33}\big|_k \frac{z^2}{2} - \epsilon_k z \Big] \frac{d\sigma_L}{dx} \quad - z\, g_k(x)$$

$$- \alpha'_z\big|_k \int_0^z \bar{g}(x,z)\, dz - \beta'_z\big|_k \int_0^z \bar{h}(x,z)\, dz \qquad (2.4.47)$$

where, as before, the function of integration, $C_k(x)$, contains all constants of integration. Let

$$\bar{\ell}_k(x,z) = -\alpha'_z\big|_k \int_0^z \bar{g}(x,z)\, dz - \beta'_z\big|_k \int_0^z \bar{h}(x,z)\, dz \qquad (2.4.48)$$

and, due to the assumed polynomial form in z for ΔT and m, we have

$$\ell_{M+1}(x,0) = 0$$

Thus, for $k = M+1$ $z=0$, we have, by definition

$$u(x,0)_{M+1} = C_{M+1}(x)$$

or

$$C_{M+1}(x) = u^\circ(x) \qquad (2.4.49)$$

Proceed to the $M+1$, $M+2$ interface, $z=H_{M+2}$, and assume continuity of $u(x,z)$ to obtain the functions of integration C_k.

$$u(x,H_{M+2})_{M+1} - u(x,H_{M+2})_{M+2} = 0$$

or

$$
\begin{aligned}
0 = {} & C_{M+1} - C_{M+2} + \Big[\left(\bar{S}'_{13}\big|_{M+1} - \bar{S}'_{13}\big|_{M+2} \right) \left(-\frac{H^2_{M+2}}{2h} + \frac{H^3_{M+2}}{h^2} \right) \\
& + \Big((\Delta + \bar{S}_{55})_{M+1} - (\Delta + \bar{S}_{55})_{M+2} \Big) \left(-\frac{H_{M+2}}{2} - \frac{H^2_{M+2}}{h} + \frac{2\,H^3_{M+2}}{h^2} \right) \\
& - H_{M+2} \left(\cancelto{0}{\delta_{M+1}} - \delta_{M+2} \right) \Big] \tau_L \\
& + \Big[\left(\bar{S}'_{13}\big|_{M+1} - \bar{S}'_{13}\big|_{M+2} \right) \left(\frac{H^2_{M+2}}{2h} + \frac{H^3_{M+2}}{h^2} \right) \\
& + \Big((\Delta + \bar{S}_{55})_{M+1} - (\Delta + \bar{S}_{55})_{M+2} \Big) \left(-\frac{H_{M+2}}{2} + \frac{H^2_{M+2}}{h} + \frac{2 H^3_{M+2}}{h^2} \right) \\
& - H_{M+2} \left(\cancelto{0}{\bar{\delta}_{M+1}} - \bar{\delta}_{M+2} \right) \Big] \tau_u \\
& + \Big[\left(\bar{S}'_{33}\big|_{M+1} - \bar{S}'_{33}\big|_{M+2} \right) \left(-\frac{H^3_{M+2}}{24} - \frac{H^4_{M+2}}{24h} + \frac{H^5_{M+2}}{20h^2} + \frac{H^2_{M+2}h}{16} \right) \\
& - H_{M+2} \left(\cancelto{0}{\eta_{M+1}} - \eta_{M+2} \right) \Big] \frac{d^2 \tau_L}{dx^2} \\
& + \Big[\left(\bar{S}'_{33}\big|_{M+1} - \bar{S}'_{33}\big|_{M+1} \right) \left(-\frac{H^3_{M+2}}{24} + \frac{H^4_{M+2}}{24h} + \frac{H^5_{M+2}}{20h^2} - \frac{H^2_{M+2}h}{16} \right) \\
& - H_{M+2} \left(\bar{\eta}_{M+1} - \bar{\eta}_4 \right) \Big] \frac{d^2 \tau_u}{dx^2} \\
& + \Big\{ \Big[(2 + 2\Delta \bar{Q}_{55})_{M+1} - (2 + 2\Delta \bar{Q}_{55})_{M+2} \Big] \left(H_{M+2} - \frac{4 H^3_{M+2}}{3h^2} \right) \\
& + \Big[(2\Delta + 2\bar{S}_{55})_{M+1}\, b_{55}\big|_{M+1} - (2\Delta + 2\bar{S}_{55})_{M+2}\, b_{55}\big|_{M+2} \Big] H_{M+2} \\
& - \frac{4}{3h^2} \left(\bar{S}'_{13} \bar{Q}_{55}\big|_{M+1} - \bar{S}'_{13} \bar{Q}_{55}\big|_{M+2} \right) H^3_{M+2} - H_{M+2} \left(\cancelto{0}{\mu_{M+1}} - \mu_{M+2} \right) \Big\} \phi \\
& + \Big\{ \left(\bar{S}'_{33} \bar{Q}_{55}\big|_{M+1} - \bar{S}'_{33} \bar{Q}_{55}\big|_{M+2} \right) \left(\frac{H^3_{M+2}}{6} - \frac{H^5_{M+2}}{15h^2} \right) \\
& - \left(\bar{S}'_{33} \bar{\alpha}\big|_{M+1} - \bar{S}'_{33} \bar{\alpha}\big|_{M+2} \right) \frac{H^2_{M+2}}{2}
\end{aligned}
$$

(over)

$$+\bar{Q}_{55}|_1\left(\bar{S}'_{33}|_{M+1}-\bar{S}'_{33}|_{M+2}\right)\frac{hH^2_{M+2}}{6}$$

$$+\left(\bar{S}'_{33}b_{55}|_{M+1}-\bar{S}'_{33}b_{55}|_{M+2}\right)\frac{H^3_{M+2}}{6}-H_{M+2}\left(\cancel{\xi_{M+1}}^{\,0}-\xi_{M+2}\right)\Big\}\frac{d^2\phi}{dx^2}$$

$$+\left[-\left(\bar{S}'_{33}|_{M+1}-\bar{S}'_{33}|_{M+2}\right)\frac{H^2_{M+2}}{2}-H_{M+2}\left(\cancel{\varepsilon_{M+1}}^{\,0}-\varepsilon_{M+2}\right)\right]\frac{dV_L}{dx}$$

$$-H_{M+2}\left(g^{(x)}_{M+1}-g^{(x)}_{M+2}\right)+\left(\bar{l}_{M+1}(x,H_{M+2})-\bar{l}_{M+2}(x,H_{M+2})\right)$$

We refer to our definition of ξ μ, η, $\bar{\eta}$, δ, $\bar{\delta}$, ε , and note that a subtraction such as $(\mu_{M+1}-\mu_{M+2})$ will cancel all but the last term in the two series, i.e. the $r = (M+2)-1$ term;

$$\mu_{M+2}-\mu_{M+3}=-b\left\{\left(\bar{S}'_{33}\bar{Q}_{55}|_{M+2}-\bar{S}'_{33}\bar{Q}_{55}|_{M+3}\right)\frac{4}{h^2}H^3_{M+3}\right\}$$

and

$$\cancel{\mu_{M+1}}^{\,0}-\mu_{M+2}=-b\left\{\left(\bar{S}'_{33}\bar{Q}_{55}|_{M+1}-\bar{S}'_{33}\bar{Q}_{55}|_{M+2}\right)\frac{4}{h^2}H^3_{M+2}\right\}$$

Combining the constants gives

$$C_k(x)=u^0(x)+D_k\frac{d^2\phi}{dx^2}+E_k\phi+F_k\frac{d^2\tau_L}{dx^2}+\bar{F}_k\frac{d^2\tau_u}{dx^2}$$
$$+G_k\tau_L+\bar{G}_k\tau_u+H'_k\frac{dV_L}{dx}+l_k(x)$$

(2.4.50)

For $k = M + 2 \rightarrow 2M + 1$

$$
\begin{aligned}
D_k = \sum_{r=M+1}^{k-1} \Big\{ & \Big[\bar{S}'_{33}|_r (\bar{Q}_{55}|_r + b_{55}|_r) - \bar{S}'_{33}|_{r+1} (\bar{Q}_{55}|_{r+1} \\
& + b_{55}|_{r+1}) \Big] \frac{H^3_{r+1}}{6} + (\bar{S}'_{33} \bar{Q}_{55}|_{r+1} - \bar{S}'_{33} \bar{Q}_{55}|_r) \frac{H^5_{r+1}}{15h^2} \\
& + (\bar{S}'_{33}|_{r+1} \bar{\alpha}_{r+1} - \bar{S}'_{33}|_r \bar{\alpha}_r) \frac{H^2_{r+1}}{2} \\
& + (\bar{S}'_{33}|_r - \bar{S}'_{33}|_{r+1}) \bar{Q}_{55}|_1 \frac{h H^2_{r+1}}{6} \\
& + b \Big[H_{r+1} (\bar{S}'_{33} \bar{Q}_{55}|_{r+1} - \bar{S}'_{33} \bar{Q}_{55}|_r) \Big(\frac{H^2_{r+1}}{2} - \frac{H^4_{r+1}}{3h^2} \Big) \\
& + H_{r+1} (\bar{S}'_{33} \bar{\alpha}|_r - \bar{S}'_{33} \bar{\alpha}|_{r+1}) H_{r+1} \\
& + H_{r+1} \bar{Q}_{55}|_1 (\bar{S}'_{33}|_{r+1} - \bar{S}'_{33}|_r) \frac{h H_{r+1}}{3} \\
& + H_{r+1} (\bar{S}'_{33} b_{55}|_{r+1} - \bar{S}'_{33} b_{55}|_r) \frac{H^2_{r+1}}{2} \Big] \Big\}
\end{aligned}
\tag{2.4.51a}
$$

$$
\begin{aligned}
D_k = \sum_{r=M+1}^{k-1} \Big\{ & (\bar{S}'_{33} \bar{Q}_{55}|_r - \bar{S}'_{33} \bar{Q}_{55}|_{r+1}) \Big(\frac{H^3_{r+1}}{6} - \frac{H^5_{r+1}}{15h^2} \cdots \\
& - b \frac{H^3_{r+1}}{2} + b \frac{H^5_{r+1}}{3h^2} \Big) + (\bar{S}'_{33}|_r b_{55}|_r - \bar{S}'_{33}|_{r+1} b_{55}|_{r+1}) \Big(\frac{H^3_{r+1}}{6} - b \frac{H^3_{r+1}}{2} \Big) \\
& + (\bar{S}'_{33}|_{r+1} \bar{\alpha}_{r+1} - \bar{S}'_{33}|_r \bar{\alpha}_r) \Big(\frac{H^2_{r+1}}{2} - b H^2_{r+1} \Big) \\
& + (\bar{S}'_{33}|_r - \bar{S}'_{33}|_{r+1}) \bar{Q}_{55}|_1 \Big(\frac{h H^2_{r+1}}{6} - b \frac{h H^2_{r+1}}{3} \Big) \Big\}
\end{aligned}
\tag{2.4.51b}
$$

$$E_k = \sum_{r=M+1}^{k-1} \Big\{ \left(\bar{S}'_{13}\,\bar{Q}_{55}\big|_{r+1} - \bar{S}'_{13}\,\bar{Q}_{55}\big|_{r} \right) \frac{4H^3_{r+1}}{3h^2}$$
$$+\left(2\Delta\bar{Q}_{55}\big|_r - 2\Delta\bar{Q}_{55}\big|_{r+1} \right) H_{r+1}$$
$$+\left(8\Delta\bar{Q}_{55}\big|_{r+1} - 8\Delta\bar{Q}_{55}\big|_{r} \right) \frac{H^3_{r+1}}{3h^2}$$
$$+\left[(2\Delta + 2\bar{S}_{55})_r\, b_{55}\big|_r - (2\Delta + 2\bar{S}_{55})_{r+1}\, b_{55}\big|_{r+1} \right] H_{r+1}$$
$$+ b\left[H_{r+1} \frac{4}{h^2} H^2_{r+1} \left(\bar{S}'_{13}\,\bar{Q}_{55}\big|_r - \bar{S}'_{13}\bar{Q}_{55}\big|_{r+1} \right) \right] \Big\} \qquad (2.4.52a)$$

$$E_k = \sum_{r=M+1}^{k-1} \Big\{ \left(\bar{S}'_{13}\,\bar{Q}_{55}\big|_{r+1} - \bar{S}'_{13}\,\bar{Q}_{55}\big|_r \right) \left(\frac{4H^3_{r+1}}{3h^2} - b\,\frac{4H^3_{r+1}}{h^2} \right)$$
$$+ \left(\Delta\bar{Q}_{55}\big|_r - \Delta\bar{Q}_{55}\big|_{r+1} \right) \left(2H_{r+1} - \frac{8H^3_{r+1}}{3h^2} \right)$$
$$+2\left[(\Delta + \bar{S}_{55})_r\, b_{55}\big|_r - (\Delta + \bar{S}_{55})_{r+1}\, b_{55}\big|_{r+1} \right] H_{r+1} \Big\} \qquad (2,4.52b)$$

$$F_k = \sum_{r=M+1}^{k-1} \Big\{ \left(\bar{S}'_{33}\big|_r - \bar{S}'_{33}\big|_{r+1} \right) \left(-\frac{H^3_{r+1}}{24} - \frac{H^4_{r+1}}{24h} + \frac{H^5_{r+1}}{20h^2} + \frac{H^2_{r+1}\,h}{16} \right)$$
$$+ b\left[H_{r+1} \left(\bar{S}'_{33}\big|_r - \bar{S}'_{33}\big|_{r+1} \right) \left(\frac{H^2_{r+1}}{8} + \frac{H^3_{r+1}}{6h} - \frac{H^4_{r+1}}{4h^2} - \frac{h\,H_{r+1}}{8} \right) \right] \Big\} \qquad (2.4.53a)$$

$$F_k = \sum_{r=M+1}^{k-1} \Big\{ \left(\bar{S}'_{33}\big|_r - \bar{S}'_{33}\big|_{r+1} \right) \left(-\frac{H^3_{r+1}}{24} + b\,\frac{H^3_{r+1}}{8} \right.$$
$$\left. - \frac{H^4_{r+1}}{24h} + b\,\frac{H^4_{r+1}}{6h} + \frac{H^5_{r+1}}{20h^2} - b\,\frac{H^5_{r+1}}{4h^2} + \frac{h\,H^2_{r+1}}{16} - b\,\frac{h\,H^2_{r+1}}{8} \right) \Big\} \qquad (2.4.53b)$$

$$\bar{F}_k = \sum_{r=M+1}^{k-1} \left\{ \left(\bar{S}'_{33}\big|_r - \bar{S}'_{33}\big|_{r+1}\right)\left(-\frac{H_{r+1}^3}{24} + \frac{H_{r+1}^4}{24h} + \frac{H_{r+1}^5}{20h^2} - \frac{hH_{r+1}^2}{16}\right) + b\left[H_{r+1}\left(\bar{S}'_{33}\big|_r - \bar{S}'_{33}\big|_{r+1}\right)\left(\frac{H_{r+1}^2}{8} - \frac{H_{r+1}^3}{6h} - \frac{H_{r+1}^4}{4h^2} + \frac{hH_{r+1}}{8}\right)\right]\right\} \tag{2.4.54a}$$

$$\bar{F}_k = \sum_{r=M+1}^{k-1} \left\{ \left(\bar{S}'_{33}\big|_r - \bar{S}'_{33}\big|_{r+1}\right)\left(-\frac{H_{r+1}^3}{24} + b\,\frac{H_{r+1}^3}{8} + \frac{H_{r+1}^4}{24h} - b\,\frac{H_{r+1}^4}{6h} + \frac{H_{r+1}^5}{20h^2} - b\,\frac{H_{r+1}^5}{4h^2} - \frac{hH_{r+1}^2}{16} + \frac{hH_{r+1}^2}{8}\right)\right\} \tag{2.4.54b}$$

$$G_k = \sum_{r=M+1}^{k-1} \left\{ \left(\bar{S}'_{13}\big|_r - \bar{S}'_{13}\big|_{r+1}\right)\left(-\frac{H_{r+1}^2}{2h} + \frac{H_{r+1}^3}{h^2}\right) + \left[(\Delta + \bar{S}_{55})_r - (\Delta + \bar{S}_{55})_{r+1}\right] \cdot \left(-\frac{H_{r+1}}{2} - \frac{H_{r+1}^2}{h} + \frac{2H_{r+1}^3}{h^2}\right) + b\left[H_{r+1}\left(\bar{S}'_{13}\big|_r - \bar{S}'_{13}\big|_{r+1}\right)\left(\frac{H_{r+1}}{h} - \frac{3H_{r+1}^2}{h^2}\right)\right]\right\} \tag{2.4.55a}$$

$$G_k = \sum_{r=M+1}^{k-1} \left\{ \left(\bar{S}'_{13}\big|_r - \bar{S}'_{13}\big|_{r+1}\right)\left(-\frac{H_{r+1}^2}{2h} + b\,\frac{H_{r+1}^2}{h} \cdots + \frac{H_{r+1}^3}{h^2} - b\,\frac{3H_{r+1}^3}{h^2}\right) + \left[(\Delta + \bar{S}_{55})_r - (\Delta + \bar{S}_{55})_{r+1}\right]\left(-\frac{H_{r+1}}{2} - \frac{H_{r+1}^2}{h} + \frac{2H_{r+1}^3}{h^2}\right)\right\} \tag{2.4.55b}$$

$$\bar{G}_k = \sum_{r=M+1}^{k-1} \Big\{ \left(\bar{S}'_{13}\big|_r - \bar{S}'_{13}\big|_{r+1} \right) \left(\frac{H^2_{r+1}}{2h} + \frac{H^3_{r+1}}{h^2} \right) + \left[(\Delta + \bar{S}_{55})_r - (\Delta + \bar{S}_{55})_{r+1} \right] \cdot$$
$$\left(\frac{-H_{r+1}}{2} + \frac{H^2_{r+1}}{h} + \frac{2H^3_{r+1}}{h^2} \right) + b\left[H_{r+1} \left(\bar{S}'_{13}\big|_r - \bar{S}'_{13}\big|_{r+1} \right) \left(-\frac{H_{r+1}}{h} - \frac{3H^2_{r+1}}{h^2} \right) \right] \Big\} \qquad (2.4.56a)$$

$$\bar{G}_k = \sum_{r=M+1}^{k-1} \Big\{ \left(\bar{S}'_{13}\big|_r - \bar{S}'_{13}\big|_{r+1} \right) \left(\frac{H^2_{r+1}}{2h} - b\,\frac{H^2_{r+1}}{h} + \frac{H^3_{r+1}}{h^2} - b\,\frac{3H^3_{r+1}}{h^2} \right)$$
$$+ \left[(\Delta + \bar{S}_{55})_r - (\Delta + \bar{S}_{55})_{r+1} \right] \left(\frac{-H_{r+1}}{2} + \frac{H^2_{r+1}}{h} + \frac{2H^3_{r+1}}{h^2} \right) \Big\} \qquad (2.4.56b)$$

$$H'_k = \sum_{r=M+1}^{k-1} \Big\{ \left(\bar{S}'_{33}\big|_{r+1} - \bar{S}'_{33}\big|_r \right) \frac{H^2_{r+1}}{2} +$$
$$b\left[H_{r+1} \left(\bar{S}'_{33}\big|_r - \bar{S}'_{33}\big|_{r+1} \right) H_{r+1} \right] \Big\} \qquad (2.4.57a)$$

$$H'_k = \sum_{r=M+1}^{k-1} \Big\{ \left(\bar{S}'_{33}\big|_{r+1} - \bar{S}'_{33}\big|_r \right) \left(\frac{H^2_{r+1}}{2} - b\, H^2_{r+1} \right) \Big\} \qquad (2.4.57b)$$

$$\ell_k(x) = \sum_{r=M+1}^{k-1} \Big\{ -H_{r+1} \left(g_r(x) - g_{r+1}(x) \right) + \bar{\ell}_r(x, H_{r+1})$$
$$- \bar{\ell}_{r+1}(x, H_{r+1}) \Big\} \qquad (2.4.58)$$

Employ continuity of u at the M, M+1 interface, $z = H_{M+1}$, to obtain similar constants for the lower laminae:

$$0 = u(x, H_{M+1})_{M+1} - u(x, H_{M+1})_{M}$$

or

$$
\begin{aligned}
0 = {} & C_{M+1}(x) - C_M(x) + \Big[\left(\bar{S}'_{13}\big|_{M+1} - \bar{S}'_{13}\big|_M \right) \left(-\frac{H^2_{M+1}}{2h} + \frac{H^3_{M+1}}{h^2} \right) \\
& + \left((\Delta + \bar{S}_{55})_{M+1} - (\Delta + \bar{S}_{55})_M \right) \left(-\frac{H_{M+1}}{2} - \frac{H^2_{M+1}}{h} + \frac{2\,H^3_{M+1}}{h^2} \right) \\
& - H_{M+1} \left(\cancelto{0}{\delta_{M+1}} - \delta_M \right) \Big] \tau_L + \Big[\left(\bar{S}'_{13}\big|_{M+1} - \bar{S}'_{13}\big|_M \right) \left(\frac{H^2_{M+1}}{2h} + \frac{H^3_{M+1}}{h^2} \right) \\
& + \left((\Delta + \bar{S}_{55})_{M+1} - (\Delta + \bar{S}_{55})_M \right) \left(-\frac{H_{M+1}}{2} + \frac{H^2_{M+1}}{h} + \frac{2\,H^3_{M+1}}{h^2} \right) \\
& - H_{M+1} \left(\cancelto{0}{\bar{\delta}_{M+1}} - \bar{\delta}_M \right) \Big] \tau_u \\
& + \Big[\left(\bar{S}'_{33}\big|_{M+1} - \bar{S}'_{33}\big|_M \right) \left(-\frac{H^3_{M+1}}{24} - \frac{H^4_{M+1}}{24h} + \frac{H^5_{M+1}}{20h^2} + \frac{H^2_{M+1} h}{16} \right) \\
& - H_{M+1} \left(\cancelto{0}{\eta_{M+1}} - \eta_M \right) \Big] \frac{d^2 \tau_L}{dx^2} \\
& + \Big[\left(\bar{S}'_{33}\big|_{M+1} - \bar{S}'_{33}\big|_M \right) \left(-\frac{H^3_{M+1}}{24} \quad \frac{H^4_{M+1}}{24h} \quad \frac{H^5_{M+1}}{20h^2} \quad \frac{H^2_{M+1} h}{16} \right) \\
& - H_{M+1} \left(\cancelto{0}{\bar{\eta}_{M+1}} - \bar{\eta}_M \right) \Big] \frac{d^2 \tau_u}{dx^2} \\
& + \Big[\left((2 + 2\Delta \bar{Q}_{55})_{M+1} - (2 + 2\Delta \bar{Q}_{55})_M \right) \left(H_{M+1} - \frac{4\,H^3_{M+1}}{3h^2} \right) \\
& + \left((2\Delta + 2\bar{S}_{55})_{M+1} b_{55}\big|_{M+1} - (2\Delta + 2\bar{S}_{55})_M b_{55}\big|_M \right) H_{M+1} \\
& - \frac{4}{3h^2} \left(\bar{S}'_{13} \bar{Q}_{55}\big|_{M+1} - \bar{S}'_{13} \bar{Q}_{55}\big|_M \right) H^3_{M+1} - H_{M+1} \left(\cancelto{0}{\mu_{M+1}} - \mu_M \right) \Big] \phi
\end{aligned}
$$

(over)

$$
\begin{aligned}
&+\Big[\left(\bar{S}'_{33}\bar{Q}_{55}\big|_{M+1}-\bar{S}'_{33}\bar{Q}_{55}\big|_{M}\right)\left(\frac{H^3_{M+1}}{6}-\frac{H^5_{M+1}}{15h^2}\right)\\
&\quad-\left(\bar{S}'_{33}\bar{\alpha}\big|_{M+1}-\bar{S}'_{33}\bar{\alpha}\big|_{M}\right)\frac{H^2_{M+1}}{2}+\bar{Q}_{55}\big|_1\left(\bar{S}'_{33}\big|_{M+1}-\bar{S}'_{33}\big|_{M}\right)\frac{hH^2_{M+1}}{6}\\
&\quad+\left(\bar{S}'_{33}b_{55}\big|_{M+1}-\bar{S}'_{33}b_{55}\big|_{M}\right)\frac{H^3_{M+1}}{6}-H_{M+1}\left(\rho_{M+1}-\rho_M\right)\Big]\frac{d^2\phi}{dx^2}\\
&+\Big[-\left(\bar{S}'_{33}\big|_{M+1}-\bar{S}'_{33}\big|_{M}\right)\frac{H^2_{M+1}}{2}-H_{M+1}\left(\cancelto{0}{\epsilon_{M+1}}-\epsilon_M\right)\Big]\frac{dv_L}{dx}\\
&-H_{M+1}\left(g_{M+1}(x)-g_M(x)\right)+\left(\bar{\ell}_{M+1}(x,H_{M+1})-\bar{\ell}_M(x,H_{M+1})\right)
\end{aligned}
$$

Thus the laminae below the mid-plane lamina have the same general constant relationships but with new limits. For lamina $k = M + 1$ the following constants are valid

$$
\begin{aligned}
D_k = \sum_{r=M+1}^{k+1}\Big\{&\Big[\bar{S}'_{33}\big|_r\left(\bar{Q}_{55}+b_{55}\right)_r-\bar{S}'_{33}\big|_{r-1}\left(\bar{Q}_{55}+b_{55}\right)_{r-1}\Big]\frac{H^3_r}{6}\\
&+\left(\bar{S}'_{33}\bar{Q}_{55}\big|_{r-1}-\bar{S}'_{33}\bar{Q}_{55}\big|_r\right)\frac{H^5_r}{15h^2}\\
&+\left(\bar{S}'_{33}\bar{\alpha}\big|_{r-1}-\bar{S}'_{33}\bar{\alpha}\big|_r\right)\frac{H^2_r}{2}\\
&+\left(\bar{S}'_{33}\big|_r-\bar{S}'_{33}\big|_{r-1}\right)\bar{Q}_{55}\big|_1\frac{hH^2_r}{6}\\
&+b\Big[H_r\left(\bar{S}'_{33}\bar{Q}_{55}\big|_{r-1}-\bar{S}'_{33}\bar{Q}_{55}\big|_r\right)\left(\frac{H^2_r}{2}-\frac{H^4_r}{3h^2}\right)\\
&\quad+H_r\left(\bar{S}'_{33}\bar{\alpha}\big|_r-\bar{S}'_{33}\bar{\alpha}\big|_{r-1}\right)H_r\\
&\quad+H_r\bar{Q}_{55}\big|_1\left(\bar{S}'_{33}\big|_{r-1}-\bar{S}'_{33}\big|_r\right)\frac{hH_r}{3}\\
&\quad+H_r\left(\bar{S}'_{33}b_{55}\big|_{r-1}-\bar{S}'_{33}b_{55}\big|_r\right)\frac{H^2_r}{2}\Big]\Big\}
\end{aligned}
\qquad (2.4.59a)
$$

$$D_k = \sum_{r=M+1}^{k+1} \Big\{ \left(\bar{S}'_{33}\, \bar{Q}_{55}|_r - \bar{S}'_{33}\bar{Q}_{55}|_{r-1} \right) \cdot \left(\frac{H_r^3}{6} - \frac{H_r^5}{15h^2} - b\frac{H_r^3}{2} + b\,\frac{H_r^5}{3h^2} \right)$$
$$+ \left(\bar{S}'_{33}\, b_{55}|_r - \bar{S}'_{33}\, b_{55}|_{r-1} \right) \left(\frac{H_r^3}{6} - b\,\frac{H_r^3}{2} \right)$$
$$+ \left(\bar{S}'_{33}|_{r-1}\, \bar{\alpha}_{r-1} - \bar{S}'_{33}|_r \bar{\alpha}_r \right) \left(\frac{H_r^2}{2} - b\, H_r^2 \right)$$
$$+ \left(\bar{S}'_{33}\,|_r - \bar{S}'_{33}|_{r-1} \right) \bar{Q}_{55}|_1 \left(\frac{h\, H_r^2}{6} - b\frac{h H_r^2}{3} \right) \Big\} \qquad (2.4.59b)$$

$$E_k = \sum_{r=M+1}^{k+1} \Big\{ \left(\bar{S}'_{13}\, \bar{Q}_{55}|_{r-1} - \bar{S}'_{13}\, \bar{Q}_{55}|_r \right) \frac{4H_r^3}{3h^2}$$
$$+ \left(2\,\Delta\bar{Q}_{55}|_r - 2\,\Delta\bar{Q}_{55}|_{r-1} \right) H_r$$
$$+ \left(8\,\Delta\,\bar{Q}_{55}|_{r-1} - 8\,\Delta\,\bar{Q}_{55}|_r \right) \frac{H_r^3}{3h^2}$$
$$+ \left((\Delta + \bar{S}_{55})_r\, b_{55}|_r - (\Delta + \bar{S}_{55})_{r-1} b_{55}|_{r-1} \right) 2H_r$$
$$+ b\left[H_r\, \frac{4}{h^2} H_r^2 \left(\bar{S}'_{13}\, \bar{Q}_{55}|_r - \bar{S}'_{13}\, \bar{Q}_{55}|_{r-1} \right) \right] \Big\} \qquad (2.4.60a)$$

$$E_k = \sum_{r=M+1}^{k+1} \Big\{ \left(\bar{S}'_{13}\bar{Q}_{55}|_{r-1} - \bar{S}'_{13}\,\bar{Q}_{55}|_r \right) \left(\frac{4H_r^3}{3h^2} - b\,\frac{4H_r^3}{h^2} \right)$$
$$+ \left(\Delta\,\bar{Q}_{55}|_r - \Delta\,\bar{Q}_{55}|_{r-1} \right) \left(2H_r - \frac{8H_r^3}{3h^2} \right)$$
$$+ \left[(\Delta + \bar{S}_{55})_r\, b_{55}|_r - (\Delta + \bar{S}_{55})_{r-1}\, b_{55}|_{r-1} \right] 2H_r \Big\} \qquad (2.4.60b)$$

$$F_k = \sum_{r=M+1}^{k+1} \Big\{ \left(\bar{S}'_{33}|_r - \bar{S}'_{33}|_{r-1} \right) \left(\frac{-H_r^3}{24} - \frac{H_r^4}{24h} + \frac{H_r^5}{20h^2} + \frac{h\,H_r^2}{16} \right)$$
$$+ b\left[H_r \left(\bar{S}'_{33}|_r - \bar{S}'_{33}|_{r-1} \right) \left(\frac{H_r^2}{8} + \frac{H_r^3}{6h} - \frac{H_r^4}{4h^2} - \frac{h\,H_r}{8} \right) \right] \Big\} \qquad (2.4.61a)$$

$$F_K = \sum_{r=M+1}^{K+1} \left\{ \left(\bar{S}'_{33}\big|_r - \bar{S}'_{33}\big|_{r-1} \right) \left(-\frac{H_r^3}{24} + b\frac{H_r^3}{8} - \frac{H_r^4}{24h} + b\frac{H_r^4}{6h} + \frac{H_r^5}{20h^2} - b\frac{H_r^5}{4h^2} + \frac{hH_r^2}{16} - b\frac{hH_r^2}{8} \right) \right\} \tag{2.4.61b}$$

$$\bar{F}_k = \sum_{r=M+1}^{k+1} \left\{ \left(\bar{S}'_{33}\big|_r - \bar{S}'_{33}\big|_{r-1} \right) \left(-\frac{H_r^3}{24} + \frac{H_r^4}{24h} + \frac{H_r^5}{20h^2} - \frac{hH_r^2}{16} \right) + b\left[H_r \left(\bar{S}'_{33}\big|_r - \bar{S}'_{33}\big|_{r-1} \right) \left(\frac{H_r^2}{8} - \frac{H_r^3}{6h} - \frac{H_r^4}{4h^2} + \frac{hH_r}{8} \right) \right] \right\} \tag{2.4.62a}$$

$$\bar{F}_k = \sum_{r=M+1}^{K+1} \left\{ \left(\bar{S}'_{33}\big|_r - \bar{S}'_{33}\big|_{r-1} \right) \left(-\frac{H_r^3}{24} + b\frac{H_r^3}{8} + \frac{H_r^4}{24h} - b\frac{H_r^4}{6h} + \frac{H_r^5}{20h^2} - b\frac{H_r^5}{4h^2} - \frac{hH_r^2}{16} + b\frac{hH_r^2}{8} \right) \right\} \tag{2.4.62b}$$

$$G_k = \sum_{r=M+1}^{k+1} \left\{ \left(\bar{S}'_{13}\big|_r - \bar{S}'_{13}\big|_{r-1} \right) \left(-\frac{H_r^2}{2h} + \frac{H_r^2}{h^2} \right) + \left[(\Delta + \bar{S}_{55})_r - (\Delta + \bar{S}_{55})_{r-1} \right] \cdot \left(-\frac{H_r}{2} - \frac{H_r^2}{h} + \frac{2H_r^3}{h^2} \right) + b\left[H_r \left(\bar{S}'_{13}\big|_r - \bar{S}'_{13}\big|_{r-1} \right) \left(\frac{H_r}{h} - \frac{3H_r^2}{h^2} \right) \right] \right\} \tag{2.4.63a}$$

$$G_k = \sum_{r=M+1}^{k+1} \left\{ \left(\bar{S}'_{13}\big|_r - \bar{S}'_{13}\big|_{r-1} \right) \left(-\frac{H_r^2}{2h} + b\frac{H_r^2}{h} + \frac{H_r^3}{h^2} - \frac{3bH_r^3}{h^2} \right) + \left[(\Delta + \bar{S}_{55})_r - (\Delta + \bar{S}_{55})_{r-1} \right] \left(-\frac{H_r}{2} - \frac{H_r^2}{h} + \frac{2H_r^3}{h^2} \right) \right\} \tag{2.4.63b}$$

$$\bar{G}_k = \sum_{r=M+1}^{k+1} \Big\{ (\bar{S}'_{13}|_r - \bar{S}'_{13}|_{r-1})\left(\frac{H_r^2}{2h} + \frac{H_r^3}{h^2}\right) + \left[(\Delta + \bar{S}_{55})_r - (\Delta + \bar{S}_{55})_{r-1}\right] \cdot$$

$$\left(\frac{-H_r}{2} + \frac{H_r^2}{h} + \frac{2H_r^3}{h^2}\right) + b\left[H_r(\bar{S}'_{13}|_r - \bar{S}'_{13}|_{r-1})\left(\frac{-H_r}{h} - \frac{3H_r^2}{h^2}\right)\right] \Big\} \tag{2.4.64a}$$

$$\bar{G}_k = \sum_{r=M+1}^{k+1} \Big\{ (\bar{S}'_{13}|_r - \bar{S}'_{13}|_{r-1})\left(\frac{H_r^2}{2h} - b\frac{H_r^2}{h} + \frac{H_r^3}{h^2} - \frac{3b\,H_r^3}{h^2}\right)$$

$$+ \left[(\Delta + \bar{S}_{55})_r - (\Delta + \bar{S}_{55})_{r-1}\right]\left(-\frac{H_r}{2} + \frac{H_r^2}{h} + \frac{2H_r^3}{h^2}\right) \Big\} \tag{2.4.64b}$$

$$H'_k = \sum_{r=M+1}^{k+1} \Big\{ (\bar{S}'_{33}|_{r-1} - \bar{S}'_{33}|_r)\frac{H_r^2}{2} + b\left[H_r(\bar{S}'_{33}|_r - \bar{S}'_{33}|_{r-1})H_r\right] \Big\} \tag{2.4.65a}$$

$$H'_k = \sum_{r=M+1}^{k+1} \Big\{ (\bar{S}'_{33}|_{r-1} - \bar{S}'_{33}|_r)\left(\frac{H_r^2}{2} - b\,H_r^2\right) \Big\} \tag{2.4.65b}$$

$$\ell_k(x) = \sum_{r=M+1}^{k+1} \Big\{ -H_r\left(g_{r-1}(x) - g_r(x)\right)$$

$$+ \left(\bar{\ell}_{r-1}(x, H_r) - \bar{\ell}_r(x, H_r)\right) \Big\} \tag{2.4.66}$$

Only the constants H_k' are symmetric about the mid-surface. All other constants - D_k, E_k, F_k, $\bar{F}_k$, G_k, $\bar{G}_k$ - are neither symmetric nor antisymmetric.

We may now write equation (2.4.47) for $u(x,z)_k$, with all functions of integration known:

$$
\begin{aligned}
u(x,z)_k = {} & u^\circ(x) - z\frac{dw^\circ}{dx} + \left[-z\, g_k(x) + \bar{\ell}_k(x,z) + \ell_k(x)\right] \\
& + \left[\bar{S}'_{13}\big|_k \left(-\frac{z^2}{2h} + \frac{z^3}{h^2}\right) + (\Delta + \bar{S}_{55})_k \left(-\frac{z}{2} - \frac{z^2}{h} + \frac{2z^3}{h^2}\right) + G_k - \delta_k z\right] \tau_L \\
& + \left[\bar{S}'_{13}\big|_k \left(\frac{z^2}{2h} + \frac{z^3}{h^2}\right) + (\Delta + \bar{S}_{55})_k \left(-\frac{z}{2} + \frac{z^2}{h} + \frac{2z^3}{h^2}\right) + \bar{G}_k - \bar{\delta}_k z\right] \tau_u \\
& + \left[\bar{S}'_{33}\big|_k \left(-\frac{z^3}{24} - \frac{z^4}{24h} + \frac{z^5}{20h^2} + \frac{hz^2}{16}\right) + F_k - \eta_k z\right] \frac{d^2\tau_L}{dx^2} \\
& + \left[\bar{S}'_{33}\big|_k \left(-\frac{z^3}{24} + \frac{z^4}{24h} + \frac{z^5}{20h^2} - \frac{hz^2}{16}\right) + \bar{F}_k - \bar{\eta}_k z\right] \frac{d^2\tau_u}{dx^2} \\
& + \Big[(2 + 2\Delta\bar{Q}_{55})_k \left(z - \frac{4z^3}{3h^2}\right) + (2\Delta + 2\bar{S}_{55})_k\, b_{55}\big|_k z \\
& \quad - \frac{4\bar{S}'_{13}\bar{Q}_{55}\big|_k}{3h^2} z^3 + E_k - \mu_k z\Big] \phi \\
& + \Big\{\bar{S}'_{33}\big|_k \Big[\bar{Q}_{55}\big|_k \left(\frac{z^3}{6} - \frac{z^5}{15h^2}\right) - \bar{\alpha}_k \frac{z^2}{2} + \bar{Q}_{55}\big|_1 \frac{hz^2}{6} \\
& \quad + b_{55}\big|_k \frac{z^3}{6}\Big] + D_k - \beta_k z\Big\} \frac{d^2\phi}{dx^2} \\
& + \left[-\bar{S}'_{33}\big|_k \frac{z^2}{2} + H'_k - \epsilon_k z\right] \frac{d\sigma_L}{dx}
\end{aligned}
\qquad (2.4.67)
$$

In what is to follow, we will, for convenience, combine our functions

$$\bar{\bar{h}}_k(x,z) = -g_k(x)\, z + \bar{\ell}_k(x,z) + \ell_k(x) \qquad (2.4.68)$$

2.5 Derivation of the Integrated Axial Load, Shear and Moment Resultants

In order to generate sufficient equations to obtain a solution for the plate element, we must derive expressions for the integrated resultants in terms of the surface tractions, the function ϕ , and the known material properties and known temperature and moisture distributions. From the constitutive relation (2.1.13), using our plane-strain assumption,

$$\sigma_x(x,z)_k = \bar{Q}_{11}\big|_k\left(\varepsilon_x - \alpha_x\Delta T - \beta_x m\right)_k + \bar{Q}_{12}\big|_k\left(\cancelto{0}{\varepsilon_y} - \alpha_y\Delta T - \beta_y m\right)_k$$
$$+\bar{Q}_{13}\big|_k\left(\varepsilon_z - \alpha_z\Delta T - \beta_z m\right) + \bar{Q}_{16}\big|_k\left(\cancelto{0}{\varepsilon_{xy}} - \alpha_{xy}\Delta T - \beta_{xy} m\right)_k \qquad (2.5.1a)$$

or, using the strain-displacement relation (2.1.20), and omitting the k subscript

$$\sigma_x = \bar{Q}_{11}\left(\frac{\partial u}{\partial x} - \alpha_x\Delta T - \beta_x m\right) + \bar{Q}_{12}\left(-\alpha_y\Delta T - \beta_y m\right)$$
$$+\bar{Q}_{13}\left(\varepsilon_z - \alpha_z\Delta T - \beta_z m\right) + \bar{Q}_{16}\left(-\alpha_{xy}\Delta T - \beta_{xy} m\right) \qquad (2.5.1b)$$

Use equation (2.4.11c) for ε_z and subtract $(\alpha_z\Delta T+\beta_z m)$ from both sides

$$\varepsilon_z - \alpha_z\Delta T - \beta_z m = \bar{S}'_{13}\sigma_x + \bar{S}'_{33}\sigma_z$$
$$+\left(\alpha'_z - \alpha_z\right)\Delta T + \left(\beta'_z - \beta_z\right)m \qquad (2.4.11d)$$

The $\bar{Q}_{13}$ term may be a second order effect. Introducing a tracing constant "c" in order to determine the coupling between axial stress σ_x and normal strain ε_z, we obtain

$$
\begin{aligned}
\sigma_x(x,z)_k = {} & \bar{Q}_{11}\big|_k\left(\frac{\partial u}{\partial x} - \alpha_x \Delta T - \beta_x m\right)_k + \bar{Q}_{12}\big|_k\left(-\alpha_y \Delta T - \beta_y m\right)_k \\
& + c\bar{Q}_{13}\big|_k \Big[\bar{S}'_{13}\big|_k \sigma_x + \bar{S}'_{33}\big|_k \sigma_z + (\alpha'_z - \alpha_z)_k \Delta T \\
& + (\beta'_z - \beta_z)_k m\Big]_k + \bar{Q}_{16}\big|_k\left(-\alpha_{xy}\Delta T - \beta_{xy} m\right)_k
\end{aligned}
\tag{2.5.1c}
$$

where σ_z may be obtained from (2.3.11) and (2.3.15), and $\frac{\partial u}{\partial x}$ from differentiating (2.4.67).

$$
\begin{aligned}
\sigma_x(x,z)_k = {} & \bar{Q}_{11}\big|_k \frac{du^\circ}{dx} - \bar{Q}_{11}\big|_k z \frac{d^2 w^\circ}{dx^2} + \bar{Q}_{11}\big|_k \frac{\partial \bar{\bar{h}}_k}{\partial x} + \Big[\bar{Q}_{11}\bar{S}'_{13}\big|_k\left(\frac{-z^2}{2h} + \frac{z^3}{h^2}\right) \\
& + \bar{Q}_{11}\big|_k\left(\Delta + \bar{S}_{55}\right)_k\left(\frac{-z}{2} - \frac{z^2}{h} + \frac{2z^3}{h^2}\right) + \bar{Q}_{11}\big|_k G_k - \bar{Q}_{11}\big|_k z\, \delta_k\Big]\frac{d\tau_L}{dx} \\
& + \Big[\bar{Q}_{11}\bar{S}'_{13}\big|_k\left(\frac{z^2}{2h} + \frac{z^3}{h^2}\right) + \bar{Q}_{11}\big|_k\left(\Delta + \bar{S}_{55}\right)_k\left(-\frac{z}{2} + \frac{z^2}{h} + \frac{2z^3}{h^2}\right) \\
& + \bar{Q}_{11}\big|_k \bar{G}_k - \bar{Q}_{11}\big|_k z\, \bar{\delta}_k\Big]\frac{d\tau_u}{dx} + \Big\{\bar{Q}_{11}\bar{S}'_{33}\big|_k \cdot \\
& \left(-\frac{z^3}{24} - \frac{z^4}{24h} + \frac{z^5}{20h^2} + \frac{hz^2}{16}\right) + \bar{Q}_{11}\big|_k F_k - \bar{Q}_{11}\big|_k z\, \eta_k\Big\}\frac{d^3\tau_L}{dx^3} \\
& + \Big\{\bar{Q}_{11}\bar{S}'_{33}\big|_k\left(-\frac{z^3}{24} + \frac{z^4}{24h} + \frac{z^5}{20h^2} - \frac{hz^2}{16}\right) + \bar{Q}_{11}\big|_k \bar{F}_k \\
& - \bar{Q}_{11}\big|_k z\, \bar{\eta}_k\Big\}\frac{d^3\tau_u}{dx^3} + \Big\{\bar{Q}_{11}\big|_k\left(2 + 2\Delta\bar{Q}_{55}\right)_k\left(z - \frac{4z^3}{3h^2}\right) \\
& + \bar{Q}_{11}\big|_k\left(2\Delta + 2\bar{S}_{55}\right)_k b_{55}\big|_k z - \bar{Q}_{11}\big|_k \frac{4\bar{S}'_{13}\bar{Q}_{55}\big|_k}{3h^2} z^3 \\
& + \bar{Q}_{11}\big|_k E_k - \bar{Q}_{11}\big|_k \mu_k z\Big\}\frac{d\phi}{dx} \\
& + \Big\{\bar{Q}_{11}\big|_k \bar{S}'_{33}\big|_k\Big[\bar{Q}_{55}\big|_k\left(\frac{z^3}{6} - \frac{z^5}{15h^2}\right) - \frac{\bar{\alpha}_k z^2}{2} \\
& + \bar{Q}_{55}\big|_1 \frac{hz^2}{6} + b_{55}\big|_k \frac{z^3}{6}\Big] + \bar{Q}_{11}\big|_k D_k - \bar{Q}_{11}\big|_k z\, \rho_k\Big\}\frac{d^3\phi}{dx^3} \\
& + \Big\{-\bar{Q}_{11}\big|_k \bar{S}'_{33}\big|_k \frac{z^2}{2} + \bar{Q}_{11}\big|_k H'_k - \bar{Q}_{11}\big|_k z\, \epsilon_k\Big\}\frac{d^2\sigma_L}{dx^2}
\end{aligned}
$$

(over)

$$- \bar{Q}_{11}|_k \alpha_x|_k \Delta T - \bar{Q}_{11}|_k \beta_x|_k m$$

$$- \bar{Q}_{12}|_k \alpha_y|_k \Delta T - \bar{Q}_{12}|_k \beta_y|_k m$$

$$+ c\, \bar{Q}_{13}|_k \Big\{ \bar{S}'_{13} \sigma_x|_k + \bar{S}'_{33}|_k \Big[-\frac{d\tau_L}{dx}\left(-\frac{z}{4} - \frac{z^2}{2h} + \frac{z^3}{h^2}\right)$$

$$- \frac{d\tau_L}{dx}\left(-\frac{z}{4} + \frac{z^2}{2h} + \frac{z^3}{h^2}\right) - \bar{Q}_{55}|_k \left(z - \frac{4z^3}{3h^2}\right)\frac{d\phi}{dx}$$

$$- b_{55}|_k\, z\, \frac{d\phi}{dx} + \left(\bar{\alpha}_k - \frac{h}{3}\bar{Q}_{55}|_1\right)\frac{d\phi}{dx} - \frac{h}{8}\frac{d\tau_L}{dx}$$

$$+ \frac{h}{8}\frac{d\tau_u}{dx} + \sigma_L \Big] + \left(\alpha'_z|_k - \alpha_z|_k\right)\Delta T$$

$$+ \left(\beta'_z|_k - \beta_z|_k\right) m \Big\}$$

$$- \bar{Q}_{16}|_k \alpha_{xy}|_k \Delta T - \bar{Q}_{16}|_k \beta_{xy}|_k m \qquad (2.5.2a)$$

Note that the secondary effect whereby σ_x appears on the right hand side of equation (2.5.2) is cancelled if $c = 0$. Grouping common terms in the above expression gives:

$$\sigma_x(x,z)_k = \frac{1}{1 - c\bar{Q}_{13}\bar{S}'_{13}|_k} \Big\{ \bar{Q}_{11}|_k \frac{du^\circ}{dx} - \bar{Q}_{11}|_k\, z\, \frac{d^2 w^\circ}{dx^2} + \bar{Q}_{11}|_k \frac{\partial \bar{\bar{h}}_x}{\partial x}$$

$$+ \Big[\bar{Q}_{11}\bar{S}'_{13}|_k \left(\frac{-z^2}{2h} + \frac{z^3}{h^2}\right) + \bar{Q}_{11}|_k (\Delta + \bar{S}_{55})_k \left(-\frac{z}{2} - \frac{z^2}{h} + \frac{2z^3}{h^2}\right)$$

$$+ \bar{Q}_{11}|_k G_k - \bar{Q}_{11}|_k\, z\, \delta_k - c\,\bar{Q}_{13}\bar{S}'_{33}\Big|_k \left(-\frac{z}{4} - \frac{z^2}{2h} + \frac{z^3}{h^2} + \frac{h}{8}\right)\Big]\frac{d\tau_L}{dx}$$

$$+ \Big[\bar{Q}_{11}\bar{S}'_{13}|_k \left(\frac{z^2}{2h} + \frac{z^3}{h^2}\right) + \bar{Q}_{11}|_k (\Delta + \bar{S}_{55})_k \left(-\frac{z}{2} + \frac{z^2}{h} + \frac{2z^3}{h^2}\right) \qquad \text{(over)}$$

$$+\bar{Q}_{11}|_k \bar{G}_k - \bar{Q}_{11}|_k z \bar{S}_k - c\bar{Q}_{13}\bar{S}'_{33}|_k\left(\frac{-z}{4}+\frac{z^2}{2h}+\frac{z^3}{h^2}-\frac{h}{8}\right)\Big]\frac{d\tau_u}{dx}$$

$$+\Big[\bar{Q}_{11}\bar{S}'_{33}|_k\left(-\frac{z^3}{24}-\frac{z^4}{24h}+\frac{z^5}{20h^2}+\frac{hz^2}{16}\right)+\bar{Q}_{11}|_k F_k$$

$$-\bar{Q}_{11}|_k z\,\eta_k\Big]\frac{d^3\tau_L}{dx^3} \quad +$$

$$\Big[\bar{Q}_{11}\bar{S}'_{33}|_k\left(-\frac{z^3}{24}+\frac{z^4}{24h}+\frac{z^5}{20h^2}-\frac{hz^2}{16}\right)+\bar{Q}_{11}|_k \bar{F}_k$$

$$-\bar{Q}_{11}|_k z\,\bar{\eta}_k\Big]\frac{d^3\tau_u}{dx^3} + \Big[\bar{Q}_{11}|_k(2+2\Delta\bar{Q}_{55})_k\left(z-\frac{4z^3}{3h^2}\right)$$

$$+\bar{Q}_{11}|_k(2\Delta+2\bar{S}_{55})_k\, b_{55}|_k z - \bar{Q}_{11}|_k\frac{4\bar{S}'_{13}\bar{Q}_{55}|_k}{3h^2}z^3$$

$$+\bar{Q}_{11}|_k E_k - \bar{Q}_{11}|_k \mu_k z - c\bar{Q}_{13}\bar{S}'_{33}|_k\Big(\bar{Q}_{55}|_k\left(z-\frac{4z^3}{3h^2}\right)$$

$$+b_{55}|_k z - \bar{\alpha}_k + \bar{Q}_{55}|_1\frac{h}{3}\Big)\Big]\frac{d\phi}{dx}$$

$$+\Big[\bar{Q}_{11}\bar{S}'_{33}|_k\Big(\bar{Q}_{55}|_k\left(\frac{z^3}{6}-\frac{z^5}{15h^2}\right)-\bar{\alpha}_k\frac{z^2}{2}+\bar{Q}_{55}|_1\frac{hz^2}{6}$$

$$+b_{55}|_k\frac{z^3}{6}\Big)+\bar{Q}_{11}|_k D_k - \bar{Q}_{11}|_k z\,\rho_k\Big]\frac{d^3\phi}{dx^3}$$

$$+\Big[-\bar{Q}_{11}\bar{S}'_{33}|_k\frac{z^2}{2}+\bar{Q}_{11}|_k H'_k - \bar{Q}_{11}|_k z\,\epsilon_k\Big]\frac{d^2 v_L}{dx^2}$$

$$+c\bar{Q}_{13}\bar{S}'_{33}|_k v_L - \bar{Q}_{11}|_k\alpha_x|_k\Delta T - \bar{Q}_{12}|_k\alpha_y|_k\Delta T$$

$$+c\bar{Q}_{13}|_k(\alpha'_z-\alpha_z)_k\Delta T - \bar{Q}_{16}|_k\alpha_{xy}|_k\Delta T - \bar{Q}_{11}|_k\beta_x|_k m$$

$$-\bar{Q}_{12}|_k\beta_y|_k m + c\bar{Q}_{13}|_k(\beta'_z-\beta_z)_k m - \bar{Q}_{16}|_k\beta_{xy}|_k m \qquad (2.5.2b)$$

We proceed to form the integrated stress and moment resultants usually defined in plate theory:

$$M_x \equiv M = \sum_{k=1}^{2M+1} \int_{H_k}^{H_{k+1}} \sigma_x(x,z)_k \; z \, dz \tag{2.5.3}$$

$$N_x \equiv N = \sum_{k=1}^{2M+1} \int_{H_k}^{H_{k+1}} \sigma_x(x,z)_k \; dz \tag{2.5.4}$$

$$Q_x \equiv Q = \sum_{k=1}^{2M+1} \int_{H_k}^{H_{k+1}} \tau_{xz}(x,z)_k \; dz \tag{2.5.5}$$

In the following integrations, terms marked ($\nearrow$) are zero under the assumption of material symmetry about the mid-plane. There are two ways this can occur:

$$n=0,1,\cdots \quad \begin{cases} \displaystyle\int_{-h/2}^{h/2} p \; z^{2n+1} \, dz = 0 & \text{for symmetric "p"} \quad p(z) = p(-z) \\[2ex] \displaystyle\int_{-h/2}^{h/2} p \; z^{2n} \, dz = 0 & \text{for anti-symmetric "p"} \quad p(z) = -p(-z) \end{cases}$$

Thus

$$M = \sum_{k=1}^{2M+1} \frac{1}{1 - c\,\bar{Q}_{13}\bar{S}'_{13}|_k} \left\{ \int_{H_k}^{H_{k+1}} \cancel{Q_{11}|_k \frac{du^\circ}{dx} z} \, dz - \int \bar{Q}_{11}|_k \frac{d^2 w^\circ}{dx^2} z^2 \, dz \right.$$

$$\left. + \int \bar{Q}_{11}|_k \frac{\partial h_k}{\partial x} z \, dz \right.$$

(over)

$$
\begin{aligned}
&+\Big[\int \bar{Q}_{11}\bar{S}'_{13}\big|_k\Big(\cancel{-\frac{z^3}{2h}}+\frac{z^4}{h^2}\Big)dz+\int \bar{Q}_{11}\big|_k(\Delta+\bar{S}_{55})_k\Big(-\frac{z^2}{2}-\cancel{\frac{z^3}{h}}+\frac{2z^4}{h^2}\Big)dz \\
&+\int \bar{Q}_{11}\big|_k G_k\, z\, dz-\int \bar{Q}_{11}\big|_k \delta_k\, z^2 dz-c\int \bar{Q}_{13}\bar{S}'_{33}\big|_k\Big(-\frac{z^2}{4}-\cancel{\frac{z^3}{2h}} \\
&+\frac{z^4}{h^2}+\cancel{\frac{hz}{8}}\Big)dz\Big]\frac{d\tau_L}{dx}+\Big[\int \bar{Q}_{11}\bar{S}'_{13}\big|_k\Big(\cancel{\frac{z^3}{2h}}+\frac{z^4}{h^2}\Big)dz \\
&+\int \bar{Q}_{11}\big|_k(\Delta+\bar{S}_{55})_k\Big(-\frac{z^2}{2}+\cancel{\frac{z^3}{h}}+\frac{2z^4}{h^2}\Big)dz+\int \bar{Q}_{11}\big|_k \bar{G}_k\, z\, dz \\
&-\int \bar{Q}_{11}\big|_k z^2\bar{\delta}_k\, dz-c\int \bar{Q}_{13}\bar{S}'_{33}\big|_k\Big(-\frac{z^2}{4}+\cancel{\frac{z^3}{2h}}+\frac{z^4}{h^2}-\cancel{\frac{hz}{8}}\Big)dz\Big]\frac{d\tau_u}{dx} \\
&+\Big[\int \bar{Q}_{11}\bar{S}'_{33}\big|_k\Big(-\frac{z^4}{24}-\cancel{\frac{z^5}{24h}}+\frac{z^6}{20h^2}+\cancel{\frac{hz^3}{16}}\Big)dz+\int \bar{Q}_{11}\big|_k F_k\, z\, dz \\
&-\int \bar{Q}_{11}\big|_k \eta_k\, z^2 dz\Big]\frac{d^3\tau_L}{dx^3}+\Big[\int \bar{Q}_{11}\bar{S}'_{33}\big|_k\Big(-\frac{z^4}{24}+\cancel{\frac{z^5}{24h}}+\frac{z^6}{20h^2}-\cancel{\frac{hz^3}{16}}\Big)dz \\
&+\int \bar{Q}_{11}\big|_k \bar{F}_k\, z\, dz-\int \bar{Q}_{11}\big|_k \bar{\eta}_k\, z^2 dz\Big]\frac{d^3\tau_u}{dx^3} \\
&+\Big[\int \bar{Q}_{11}\big|_k(2+2\Delta\bar{Q}_{55})_k\Big(z^2-\frac{4z^4}{3h^2}\Big)dz \\
&\quad+\int \bar{Q}_{11}\big|_k(2\Delta+2\bar{S}_{55})_k\, b_{55}\big|_k\, z^2 dz-\int \bar{Q}_{11}\big|_k\frac{4\bar{S}'_{13}\bar{Q}_{55}\big|_k}{3h^2}z^4 dz \\
&\quad+\int \bar{Q}_{11}\big|_k E_k\, z\, dz-\int \bar{Q}_{11}\big|_k \mu_k\, z^2 dz \\
&\quad+c\int \bar{Q}_{13}\bar{S}'_{33}\big|_k\Big(-\bar{Q}_{55}\big|_k z^2+\bar{Q}_{55}\big|_k\frac{4z^4}{3h^2}-b_{55}\big|_k z^2 \\
&\qquad+\bar{\alpha}_k z-\cancelto{0}{\bar{Q}_{55}\big|_1\frac{hz}{3}}\Big)dz\Big]\frac{d\phi}{dx} \\
&+\Big[\int \bar{Q}_{11}\bar{S}'_{33}\big|_k\Big(\bar{Q}_{55}\big|_k\frac{z^4}{6}-\bar{Q}_{55}\big|_k\frac{z^6}{15h^2}-\bar{\alpha}_k\frac{z^3}{2} \\
&\quad+\cancel{\bar{Q}_{55}\big|_1\frac{hz^3}{6}}+b_{55}\big|_k\frac{z^4}{6}\Big)dz+\int \bar{Q}_{11}\big|_k D_k\, z\, dz
\end{aligned}
$$

(over)

$$-\int \bar{Q}_{11}|_k \rho_k \, z^2 dz \Big] \frac{d^3\phi}{dx^3}$$

$$+\Big[-\int \bar{Q}_{11}|_k \bar{S}'_{33}|_k \frac{z^3}{2} dz + \int \bar{Q}_{11} H'_k z \, dz$$

$$-\int \bar{Q}_{11}|_k \epsilon_k z^2 dz \Big] \frac{d^2 \bar{v}_L}{dx^2} + c \int \bar{Q}_{13} \bar{S}'_{33}|_k \bar{v}_L z \, dz$$

$$-\int \left(\bar{Q}_{11} \alpha_x|_k + \bar{Q}_{12} \alpha_y|_k - c \bar{Q}_{13}|_k (\alpha'_z - \alpha_z)_k + \bar{Q}_{16} \alpha_{xy}|_k \right) \Delta T z \, dz$$

$$-\int \left(\bar{Q}_{11} \beta_x|_k + \bar{Q}_{12} \beta_y|_k - c \bar{Q}_{13}|_k (\beta'_z - \beta_z)_k + \bar{Q}_{16} \alpha_{xy}|_k \right) m z \, dz \Big\}$$

Employing a compact notation and evaluating the integrals gives:

$$M = -D_{11} \frac{d^2 w^0}{dx^2} + F \frac{d^3\phi}{dx^3} + G \frac{d\phi}{dx} + H \frac{d^3 \tau_L}{dx^3}$$

$$+ I \frac{d\tau_L}{dx} + \bar{H} \frac{d^3 \tau_u}{dx^3} + \bar{I} \frac{d\tau_u}{dx}$$

$$+ \bar{h}(x) - M^T_{(x)} - M^m_{(x)} \qquad (2.5.6)$$

where

$$D_{11} = \sum_{k=1}^{2N+1} \frac{1}{1 - c \bar{Q}_{13} \bar{S}'_{13}|_k} \left\{ \bar{Q}_{11}|_k \left(\frac{H^3_{k+1} - H^3_k}{3} \right) \right\} \qquad (2.5.7)$$

$$I = \sum_{k=1}^{2M+1} \frac{1}{1-c\bar{Q}_{13}\bar{S}'_{13}|_k} \Big\{ \bar{Q}_{11}\bar{S}'_{13}|_k \left(\frac{H_{k+1}^5 - H_k^5}{5h^2}\right)$$
$$+ \bar{Q}_{11}|_k(\Delta+\bar{S}_{55})_k \left[-\frac{(H_{k+1}^3 - H_k^3)}{6} + \frac{2(H_{k+1}^5 - H_k^5)}{5h^2}\right]$$
$$+ \bar{Q}_{11}|_k G_k \left(\frac{H_{k+1}^2 - H_k^2}{2}\right) - \bar{Q}_{11}|_k \delta_k \left(\frac{H_{k+1}^3 - H_k^3}{3}\right)$$
$$- c\,\bar{Q}_{13}\bar{S}'_{33}|_k \left[-\frac{(H_{k+1}^3 - H_k^3)}{12} + \frac{(H_{k+1}^5 - H_k^5)}{5h^2}\right]\Big\} \tag{2.5.8}$$

$$\bar{I} = \sum_{k=1}^{2M+1} \frac{1}{1-c\bar{Q}_{13}\bar{S}'_{13}|_k} \Big\{ \bar{Q}_{11}\bar{S}'_{13}|_k \left(\frac{H_{k+1}^5 - H_k^5}{5h^2}\right) + \bar{Q}_{11}|_k(\Delta+\bar{S}_{55})_k \,\bullet$$
$$\left[-\frac{(H_{k+1}^3 - H_k^3)}{6} + \frac{2(H_{k+1}^5 - H_k^5)}{5h^2}\right] + \bar{Q}_{11}|_k \bar{G}_k \left(\frac{H_{k+1}^2 - H_k^2}{2}\right) - \bar{Q}_{11}|_k \bar{\delta}_k \left(\frac{H_{k+1}^3 - H_k^3}{3}\right)$$
$$- c\bar{Q}_{13}\bar{S}'_{33}|_k \left[-\frac{(H_{k+1}^3 - H_k^3)}{12} + \frac{(H_{k+1}^5 - H_k^5)}{5h^2}\right]\Big\} \tag{2.5.9}$$

$$H = \sum_{k=1}^{2M+1} \frac{1}{1-c\bar{Q}_{13}\bar{S}'_{13}|_k} \Big\{ \bar{Q}_{11}\bar{S}'_{33}|_k \left[-\frac{(H_{k+1}^5 - H_k^5)}{120} + \frac{(H_{k+1}^7 - H_k^7)}{140h^2}\right]$$
$$+ \bar{Q}_{11}|_k F_k \frac{(H_{k+1}^2 - H_k^2)}{2} - \bar{Q}_{11}|_k \eta_k \frac{(H_{k+1}^3 - H_k^3)}{3} \Big\} \tag{2.5.10}$$

$$\bar{H} = \sum_{k=1}^{2M+1} \frac{1}{1-c\bar{Q}_{13}\bar{S}'_{13}|_k} \Big\{ \bar{Q}_{11}\bar{S}'_{33}|_k \left[-\frac{(H_{k+1}^5 - H_k^5)}{120} + \frac{(H_{k+1}^7 - H_k^7)}{140h^2}\right]$$
$$+ \bar{Q}_{11}|_k \bar{F}_k \frac{(H_{k+1}^2 - H_k^2)}{2} - \bar{Q}_{11}|_k \bar{\eta}_k \frac{(H_{k+1}^3 - H_k^3)}{3} \Big\} \tag{2.5.11}$$

$$G = \sum_{k=1}^{2M+1} \frac{1}{1 - c\,\overline{Q}_{13}\,\overline{S}'_{13}|_k} \Big\{ \overline{Q}_{11}|_k \,(2 + 2\Delta\overline{Q}_{55})_k \Big[\frac{(H^3_{k+1} - H^3_k)}{3} - \frac{4(H^5_{k+1} - H^5_k)}{15h^2} \Big]$$
$$+ \overline{Q}_{11}|_k \,(2\Delta + 2\overline{S}_{55})_k \, b_{55}|_k \, \frac{(H^3_{k+1} - H^3_k)}{3}$$
$$- \overline{Q}_{11}|_k \, \frac{4\,\overline{S}'_{13}\overline{Q}_{55}|_k}{3h^2} \, \frac{(H^5_{k+1} - H^5_k)}{5} + \overline{Q}_{11}|_k \, E_k \, \frac{(H^2_{k+1} - H^2_k)}{2}$$
$$- \overline{Q}_{11}|_k \, \mu_k \, \frac{(H^3_{k+1} - H^3_k)}{3} + c\,\overline{Q}_{13}\,\overline{S}'_{33}|_k \Big[-\overline{Q}_{55}|_k \, \frac{(H^3_{k+1} - H^3_k)}{3}$$
$$+ \overline{Q}_{55}|_k \, \frac{4}{3h^2} \, \frac{(H^5_{k+1} - H^5_k)}{5} - b_{55}|_k \, \frac{(H^3_{k+1} - H^3_k)}{3} + \overline{\alpha}_k \, \frac{(H^2_{k+1} - H^2_k)}{2} \Big] \Big\} \qquad (2.5.12)$$

$$\overline{h}(x) = \sum_{k=1}^{2M+1} \frac{1}{1 - c\,\overline{Q}_{13}\,\overline{S}'_{13}|_k} \Big\{ \overline{Q}_{11}|_k \, \frac{\partial \overline{\overline{h}}_k}{\partial x} \, \frac{(H^2_{k+1} - H^2_k)}{2} \Big\} \qquad (2.5.13)$$

$$F = \sum_{k=1}^{2M+1} \frac{1}{1 - c\,\overline{Q}_{13}\,\overline{S}'_{13}|_k} \Big\{ \overline{Q}_{11}\,\overline{S}'_{33}\big)_k \Big[\overline{Q}_{55}|_k \, \frac{(H^5_{k+1} - H^5_k)}{30}$$
$$- \overline{Q}_{55}|_k \, \frac{(H^7_{k+1} - H^7_k)}{(15)(7)h^2} - \overline{\alpha}_k \, \frac{(H^4_{k+1} - H^4_k)}{8} + b_{55}|_k \, \frac{(H^5_{k+1} - H^5_k)}{30} \Big]$$
$$+ \overline{Q}_{11}|_k \, D_k \, \frac{(H^2_{k+1} - H^2_k)}{2} - \overline{Q}_{11}|_k \, \rho_k \, \frac{(H^3_{k+1} - H^3_k)}{3} \Big\} \qquad (2.5.14)$$

$$M^T = \sum_{k=1}^{2M+1} \frac{1}{1 - c\,\overline{Q}_{13}\,\overline{S}'_{13}|_k} \Big\{ \big(\overline{Q}_{11}|_k \, \alpha_x|_k + \overline{Q}_{12}|_k \, \alpha_y|_k$$
$$- c\,\overline{Q}_{13}|_k \, (\alpha'_z - \alpha_z)_k + \overline{Q}_{16}|_k \, \alpha_{xy}|_k \big) \int_{H_k}^{H_{k+1}} \Delta T z \, dz \Big\} \qquad (2.5.15)$$

$$M^m = \sum_{k=1}^{2M+1} \frac{1}{1-c\bar{Q}_{13}\bar{S}'_{13}|_k} \Big\{ \Big(\bar{Q}_{11}|_k \beta_x|_k + \bar{Q}_{12}|_k \beta_y|_k - c\bar{Q}_{13}|_k (\beta'_z - \beta_z)_k + \bar{Q}_{16}|_k \beta_{xy}|_k \Big) \int_{H_k}^{H_{k+1}} m\, z\, dz \Big\} \qquad (2.5.16)$$

Certain simplications of these equations are important:

1) If b=0, J=0
2) If $\Delta T = \Delta T(x)$, $M^T = 0$
   If $m = m(x)$, $M^m = 0$
3) If $\Delta T = \Delta T(z)$, $m = m(z)$, $\bar{h}(x) = 0$ (2.5.17)

Next, we evaluate the integrals for N:

$$N = \sum_{k=1}^{2M+1} \frac{1}{1-c\bar{Q}_{13}\bar{S}'_{13}|_k} \Big\{ \int_{H_k}^{H_{k+1}} \bar{Q}_{11}|_k \frac{du^o}{dx} dz - \int \bar{Q}_{11}|_k \cancel{\frac{d^2 w^o}{dx^2}} z\, dz$$

$$+ \int \bar{Q}_{11}|_k \frac{\partial \bar{\bar{h}}_k}{\partial x} dz$$

$$+ \Big[\int \bar{Q}_{11}\bar{S}'_{13}|_k \Big(-\frac{z^2}{2h} + \cancel{\frac{z^3}{h^2}}\Big) dz + \int \bar{Q}_{11}|_k (\Delta + \bar{S}_{55})_k \Big(\cancel{-\frac{z}{2}} - \frac{z^2}{2h} + \cancel{\frac{2z^3}{h^2}}\Big) dz$$

$$+ \int \bar{Q}_{11}|_k G_k\, dz - \int \bar{Q}_{11}|_k \delta_k z\, dz - c\int \bar{Q}_{13}\bar{S}'_{33}|_k \Big(\cancel{-\frac{z}{4}} - \frac{z^2}{2h} + \cancel{\frac{z^3}{h^2}} + \frac{h}{8}\Big) dz \Big] \frac{d\tau_l}{dx}$$

$$+ \Big[\int \bar{Q}_{11}\bar{S}'_{13}|_k \Big(\frac{z^2}{2h} + \cancel{\frac{z^3}{h^2}}\Big) dz + \int \bar{Q}_{11}|_k (\Delta + \bar{S}_{55})_k \Big(\cancel{-\frac{z}{2}} + \frac{z^2}{h} + \cancel{\frac{2z^3}{h^2}}\Big) dz$$

$$+ \int \bar{Q}_{11}|_k \bar{G}_k\, dz - \int \bar{Q}_{11}|_k \bar{\delta}_k z\, dz$$

$$- c\int \bar{Q}_{13}\bar{S}'_{33}|_k \Big(\cancel{-\frac{z}{4}} + \frac{z^2}{2h} + \cancel{\frac{z^3}{h^2}} - \frac{h}{8}\Big) dz \Big] \frac{d\tau_u}{dx}$$

(over)

$$+\Big[\ \bar{Q}_{11}\bar{S}'_{33}|_k\left(\cancel{\frac{-z^3}{24}}-\frac{z^4}{24h}+\cancel{\frac{z^5}{20h^2}}+\frac{hz^2}{16}\right)dz$$

$$+\int\bar{Q}_{11}|_k\,F_k\,dz-\int\bar{Q}_{11}|_k\,\eta_k\,z\,dz\Big]\frac{d^3\tau_L}{dx^3}$$

$$+\Big[\int\bar{Q}_{11}\bar{S}'_{33}|_k\left(\cancel{\frac{-z^3}{24}}+\frac{z^4}{24h}+\cancel{\frac{z^5}{20h^2}}-\frac{hz^2}{16}\right)dz$$

$$+\int\bar{Q}_{11}|_k\,\bar{F}_k\,dz-\int\bar{Q}_{11}|_k\,z\,\bar{\eta}_k\,dz\Big]\frac{d^3\tau_u}{dx^3}$$

$$+\Big[\int\bar{Q}_{11}|_k\,(2+2\Delta\bar{Q}_{55})_k\left(\cancel{z-\frac{4z^3}{3h^2}}\right)dz$$

$$+\int\bar{Q}_{11}|_k\,(2\Delta+2\bar{S}_{55})_k\,\cancel{b_{55}|_k\,z\,dz}$$

$$-\int\bar{Q}_{11}|_k\,\cancel{\frac{4\bar{S}'_{13}\bar{Q}_{55}|_k}{3h^2}z^3\,dz}+\int\bar{Q}_{11}|_k\,\cancel{E_k\,dz}$$

$$-\int\bar{Q}_{11}|_k\,\cancel{\mu_k\,z\,dz}+c\int\bar{Q}_{13}\bar{S}'_{33}|_k\Big(-\cancel{\bar{Q}_{55}|_k\,z}$$

$$+\bar{Q}_{55}|_k\cancel{\frac{4z^3}{3h^2}}-\cancel{b_{55}|_k\,z}+\bar{\alpha}_k-\frac{h}{3}\bar{Q}_{55}|_1\Big)dz\Big]\frac{d\phi}{dx}$$

$$+\Big[\int\bar{Q}_{11}\bar{S}'_{33}|_k\Big(\cancel{\bar{Q}_{55}|_k\frac{z^3}{6}}-\cancel{\bar{Q}_{55}|_k\frac{z^5}{15h^2}}-\bar{\alpha}_k\frac{z^2}{2}$$

$$+\bar{Q}_{55}|_1\frac{hz^2}{6}+\cancel{b_{55}|_k\frac{z^3}{6}}\Big)dz+\int\bar{Q}_{11}|_k\,D_k\,dz$$

$$-\int\bar{Q}_{11}|_k\,z\,\rho_k\,dz\Big]\frac{d^3\phi}{dx^3}$$

$$+\Big[-\int\bar{Q}_{11}\bar{S}'_{33}|_k\frac{z^2}{2}dz+\int\bar{Q}_{11}|_k\,H'_k\,dz$$

$$-\int\bar{Q}_{11}|_k\,z\,\epsilon_k\,dz\Big]\frac{d^2v_L}{dx^2}+c\int\bar{Q}_{13}\bar{S}'_{33}|_k\,v_L\,dz$$

$$-\int\left(\bar{Q}_{11}\alpha_x|_k+\bar{Q}_{12}\alpha_y|_k-c\bar{Q}_{13}|_k(\alpha'_z-\alpha_z)_k+\bar{Q}_{16}\alpha_{xy}|_k\right)\Delta T\,dz$$ (over)

$$-\int \left(\overline{Q}_{11}\, \beta_x \big|_k + \overline{Q}_{12}\, \beta_y \big|_k - c\, \overline{Q}_{13}\big|_k (\beta_z' - \beta_z)_k + \overline{Q}_{16}\, \beta_{xy}\big|_k \right) m\, dz$$

Again employing simplified notation and evaluating the integrals gives:

$$N = A \frac{d^3\phi}{dx^3} + B \frac{d\phi}{dx} + C \frac{d^3\tau_L}{dx^3} + D \frac{d\tau_L}{dx} + \overline{C} \frac{d^3\tau_u}{dx^3} + \overline{D} \frac{d\tau_u}{dx} + E \frac{d^2 v_L}{dx^2} + \overline{A} \frac{du^0}{dx} - N^T - N^m + \overline{E}\, v_L + h^*(x) \tag{2.5.18}$$

where

$$A = \sum_{k=1}^{2M+1} \frac{1}{1 - c\, \overline{Q}_{13} \overline{S}_{13}'\big|_k} \left\{ \overline{Q}_{11} \overline{S}_{33}'\big|_k \left[-\overline{\alpha}_k \frac{(H_{k+1}^3 - H_k^3)}{6} + Q_{11}\big|_1 \frac{h(H_{k+1}^3 - H_k^3)}{18} \right] + \overline{Q}_{11}\big|_k D_k (H_{k+1} - H_k) - \overline{Q}_{11}\big|_k \rho_k \frac{(H_{k+1}^2 - H_k^2)}{2} \right\} \tag{2.5.19}$$

$$B = \sum_{k=1}^{2M+1} \frac{1}{1 - c\, \overline{Q}_{13} \overline{S}_{13}'\big|_k} \left\{ \overline{Q}_{11}\big|_k E_k (H_{k+1} - H_k) + c\, \overline{Q}_{13} \overline{S}_{33}'\big|_k \left(\overline{\alpha}_k - \frac{h}{3} \overline{Q}_{55}\big|_1 \right) (H_{k+1} - H_k) \right\} \tag{2.5.20}$$

$$C = \sum_{k=1}^{2M+1} \frac{1}{1-c\,\overline{Q}_{13}\overline{S}'_{13}|_k} \Big\{ \overline{Q}_{11}\overline{S}'_{33}|_k \Big[-\frac{(H_{k+1}^5 - H_k^5)}{120\,h} + \frac{h(H_{k+1}^3 - H_k^3)}{48} \Big]$$
$$+ \overline{Q}_{11}|_k\, F_k\, (H_{k+1} - H_k) - \overline{Q}_{11}|_k\, \eta_k\, \frac{(H_{k+1}^2 - H_k^2)}{2} \Big\} \qquad (2.5.21)$$

$$D = \sum_{k=1}^{2M+1} \frac{1}{1-c\,\overline{Q}_{13}\overline{S}'_{13}|_k} \Big\{ -\overline{Q}_{11}\overline{S}'_{33}|_k\, \frac{(H_{k+1}^3 - H_k^3)}{6h}$$
$$- \overline{Q}_{11}|_k\, (\Delta + \overline{S}_{55})_k\, \frac{(H_{k+1}^3 - H_k^3)}{3h} + \overline{Q}_{11}|_k\, G_k\, (H_{k+1} - H_k)$$
$$- \overline{Q}_{11}|_k\, \delta_k\, \frac{(H_{k+1}^2 - H_k^2)}{2} + c\,\overline{Q}_{13}\overline{S}'_{33}|_k \Big[\frac{(H_{k+1}^3 - H_k^3)}{6h} - \frac{h(H_{k+1} - H_k)}{8} \Big] \Big\} \qquad (2.5.22)$$

$$\overline{C} = \sum_{k=1}^{2M+1} \frac{1}{1-c\,\overline{Q}_{13}\overline{S}'_{13}|_k} \Big\{ \overline{Q}_{11}\overline{S}'_{33}|_k \Big[\frac{(H_{k+1}^5 - H_k^5)}{120h} - \frac{h(H_{k+1}^3 - H_k^3)}{48} \Big]$$
$$+ \overline{Q}_{11}|_k\, \overline{F}_k\, (H_{k+1} - H_k) - \overline{Q}_{11}|_k\, \overline{\eta}_k\, \frac{(H_{k+1}^2 - H_k^2)}{2} \Big\} \qquad (2.5.23)$$

$$\overline{D} = \sum_{k=1}^{2M+1} \frac{1}{1-c\,\overline{Q}_{13}\overline{S}'_{13}|_k} \Big\{ \overline{Q}_{11}|_k\, \overline{S}'_{13}|_k\, \frac{(H_{k+1}^3 - H_k^3)}{6h}$$
$$+ \overline{Q}_{11}|_k\, (\Delta + \overline{S}_{55})_k\, \frac{(H_{k+1}^3 - H_k^3)}{3h} + \overline{Q}_{11}|_k\, \overline{G}_k\, (H_{k+1} - H_k)$$
$$- \overline{Q}_{11}|_k\, \overline{\delta}_k\, \frac{(H_{k+1}^2 - H_k^2)}{2} - c\,\overline{Q}_{13}\overline{S}'_{33}|_k \Big[\frac{(H_{k+1}^3 - H_k^3)}{6h} - \frac{h(H_{k+1} - H_k)}{8} \Big] \Big\} \qquad (2.5.24)$$

$$E = \sum_{k=1}^{2M+1} \frac{1}{1-c\bar{Q}_{13}\bar{S}'_{13}|_k} \Big\{ -\bar{Q}_{11}\ \bar{S}'_{33}|_k \frac{(H_{k+1}^3 - H_k^3)}{6}$$

$$+ \bar{Q}_{11}|_k H'_k (H_{k+1} - H_k) - \bar{Q}_{11}|_k \epsilon_k \frac{(H_{k+1}^2 - H_k^2)}{2} \Big\} \qquad (2.5.25)$$

$$\bar{A} = \sum_{k=1}^{2M+1} \frac{1}{1-c\bar{Q}_{13}\bar{S}'_{13}|_k} \Big\{ \bar{Q}_{11}|_k (H_{k+1} - H_k) \Big\} \qquad (2.5.26)$$

$$N^T = \sum_{k=1}^{2M+1} \frac{1}{1-c\bar{Q}_{13}\bar{S}'_{13}|_k} \Big\{ \big(\bar{Q}_{11}|_k \alpha_x|_k + \bar{Q}_{12}|_k \alpha_y|_k$$

$$- c\bar{Q}_{13}|_k (\alpha'_z - \alpha_z)_k + \bar{Q}_{16}|_k \alpha_{xy}|_k \big) \int_{H_k}^{H_{k+1}} \Delta T\, dz \Big\} \qquad (2.5.27)$$

$$N^m = \sum_{k=1}^{2M+1} \frac{1}{1-c\bar{Q}_{13}\bar{S}'_{13}|_k} \Big\{ \big(\bar{Q}_{11}|_k \beta_x|_k + \bar{Q}_{12}|_k \beta_y|_k$$

$$- c\bar{Q}_{13}|_k (\beta'_z - \beta_z)_k + \bar{Q}_{16}|_k \beta_{xy}|_k \big) \int_{H_k}^{H_{k+1}} m\, dz \Big\} \qquad (2.5.28)$$

$$\bar{E} = \sum_{k=1}^{2M+1} \frac{1}{1 - c\,\bar{Q}_{13}\bar{S}'_{13}\big|_k} \left\{ c\,\bar{Q}_{13}\bar{S}'_{13}\big|_k (H_{k+1} - H_k) \right\} \qquad (2.5.29)$$

$$h^*_{(x)} = \sum_{k=1}^{2M+1} \frac{1}{1 - c\,\bar{Q}_{13}\bar{S}'_{13}\big|_k} \left\{ \bar{Q}_{11}\big|_k \frac{\partial \bar{\bar{h}}_k}{\partial x} (H_{k+1} - H_k) \right\} \qquad (2.5.30)$$

Certain simplications of these equations are important:

1) If $c = 0$, $\bar{E} = 0$

2) If $\Delta T = \Delta T(x)$, $m = m(x)$, then $h(x) = 0$.

Finally, to determine Q , use the form (2.3.1) for $\tau_{xz}\big|_k$ with (2.3.2), (2.3.3), and (2.3.4) in equation (2.5.5):

$$Q = \sum_{k=1}^{2M+1} \left\{ \int_{H_k}^{H_{k+1}} \tau_L \left(-\frac{1}{4} - \cancelto{0}{\frac{z}{h}} + \frac{3z^2}{h^2}\right) dz \right.$$
$$+ \int \tau_u \left(-\frac{1}{4} + \cancelto{0}{\frac{z}{h}} + \frac{3z^2}{h^2}\right) dz + \left[\int \bar{Q}_{55}\big|_k \left(1 - \frac{4z^2}{h^2}\right) dz \right.$$
$$\left. \left. + \int b_{55}\big|_k \, dz \right] \phi \right\}$$

or

$$Q = k_5 \phi \qquad (2.5.31)$$

where

$$k_5 = \sum_{k=1}^{2M+1} \left\{ \bar{Q}_{55}\big|_k (H_{k+1} - H_k) - \frac{4\bar{Q}_{55}\big|_k}{h^2} \frac{(H_{k+1}^3 - H_k^3)}{3} + b_{55}\big|_k (H_{k+1} - H_k) \right\} \tag{2.5.32}$$

The simplified form of equation (2.5.31) is due to the convenient forms chosen for $f_1(z)$, $f_2(z)$; e.g.

$$\sum_{k=1}^{2M+1} \int_{H_k}^{H_{k+1}} \tau_{L(x)} \left(-\frac{1}{4} - \frac{z}{h} + \frac{3z^2}{h^2} \right) dz = \tau_L \left[-\frac{z}{4} + \frac{z^3}{h^2} \right]_{-h/2}^{h/2}$$

$$\equiv 0$$

2.6 Derivation of Equilibrium Equations for Integrated Resultants

At this juncture, we will derive the equilibrium equations for the plate element subjected to surface shear and normal stresses. This is done to obtain equations relating the integrated stress couple and resultants to the surface tractions.

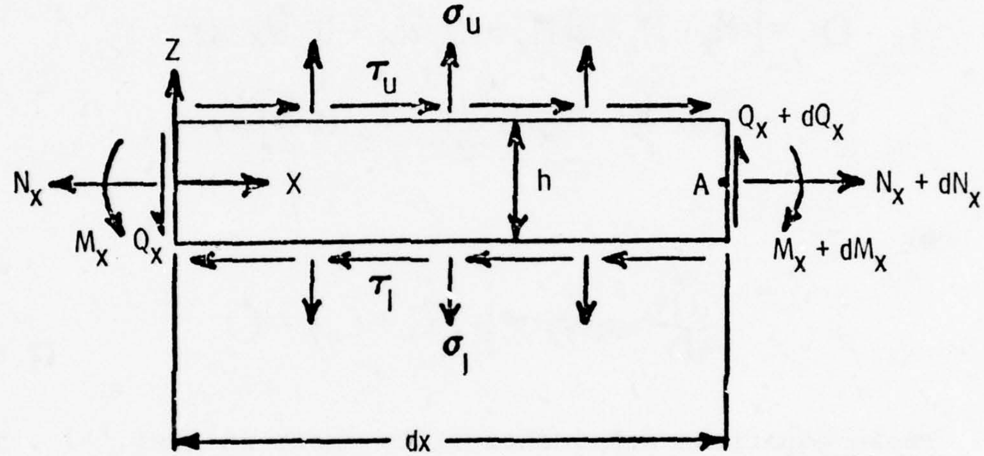

FIGURE 7

Differential Force and Moment Components

Assuming a static loading condition,

$$\Sigma F_x = 0 = [N_x + dN_x - N_x - \tau_L\, dx + \tau_u\, dx]\, dy$$

or

$$\frac{dN_x}{dx} - \tau_L + \tau_u = 0 \qquad (2.6.1)$$

$$\Sigma F_y = 0 = \left[-Q_x + Q_x + dQ_x - \sigma_L dx + \sigma_u dx\right] dy$$

or

$$\frac{dQ_x}{dx} - \sigma_L + \sigma_u = 0 \tag{2.6.2}$$

$$\Sigma M_A = 0 \qquad M + \curvearrowleft$$

$$0 = \Big[M_x - M_x - dM_x + Q_x\,dx + \sigma_L\,dx\,\frac{dx}{2} - \tau_L\,dx\,\frac{h}{2} - \sigma_u\,dx\,\frac{dx}{2} - \tau_u\,dx\,\frac{h}{2}\Big] dy$$

or

$$\frac{dM_x}{dx} - Q_x + \frac{h}{2}\left(\tau_L + \tau_u\right) = 0 \tag{2.6.3}$$

These equations are identical to those in [Ref. 5] , p. 10, with all terms relating to $\frac{\partial}{\partial y}$ of stresses set to zero pursuant to the plane-strain approximation.

Should the loads be compressive so that large deflection or buckling results, the normally second order effect of the vertical component of the in-plane force resultant $N_x \equiv N$ becomes important. The pertinent equation in this case is:

$$\sum F_z = 0 = \Big[-Q_x + Q_x + dQ_x - \tau_L\, dx + \tau_u\, dx + N_x \frac{\partial^2 w^\circ}{\partial x^2}\, dx + N_x \frac{\partial w^\circ}{\partial x} + \frac{\partial N_x}{\partial x} \frac{\partial w^\circ}{\partial x}\, dx - N_x \frac{\partial w^\circ}{\partial x} \Big]\, dy$$

or

$$0 = \frac{dQ_x}{dx} - \tau_L + \tau_u + \left(N_x \frac{\partial^2 w^\circ}{\partial x^2} + \frac{\partial N_x}{\partial x} \frac{\partial w^\circ}{\partial x} \right)$$

For further information, see [Ref. 5] p. 78.

2.7 Coefficients for a Specific Hygrothermal Case

We will investigate the case for which the thermal and moisture effects may be expressed in the following forms, which are assumed to be sufficiently general for all practical cases of interest.

$$\Delta T_{(x,z)} = (\Lambda_1 + \Lambda_2 x)(\Lambda_3 + \Lambda_4 z + \Lambda_5 z^2 + \Lambda_6 z^3) \qquad (2.7.1)$$

$$m(x,z) = (\Gamma_1 + \Gamma_2 x)(\Gamma_3 + \Gamma_4 z + \Gamma_5 z^2 + \Gamma_6 z^3) \qquad (2.7.2)$$

In all the equations which follow, the original definition is given first, with the equation number, and is immediately followed by the simplified form which the definition takes if we use (2.7.1) and (2.7.2).

The functions defined in equations (2.4.23) and (2.4.24) take the form

$$\bar{g}(x,z) = \int_0^z \frac{\partial}{\partial x} \Delta T \, dz$$

$$= \Lambda_2 \left(\Lambda_3 z + \frac{\Lambda_4 z^2}{2} + \frac{\Lambda_5 z^3}{3} + \frac{\Lambda_6 z^4}{4} \right) \qquad (2.4.23)$$

$$\bar{h}(x,z) = \int_0^z \frac{\partial}{\partial x} m \, dz$$

$$= \Gamma_2 \left(\Gamma_3 z + \frac{\Gamma_4 z^2}{2} + \frac{\Gamma_5 z^3}{3} + \frac{\Gamma_6 z^4}{4} \right) \qquad (2.4.24)$$

Now, proceed to the next step, equations (2.4.27) and (2.4.41). For k = M + 2 → 2M + 1 , equation (2.4.33) applies:

$$g_k(x) = b\sum_{k=M+1}^{k-1}\left\{\bar{g}\,(x, H_{r+1})\left[\alpha'_z|_r - \alpha'_z|_{r+1}\right] + \bar{h}\,(x, H_{r+1})\left[\beta'_z|_r - \beta'_z|_{r+1}\right]\right\} \quad (2.4.33)$$

or

$$g_k(x) = g_k = b\sum_{r=M+1}^{k-1}\left\{\Lambda_2\left(\Lambda_3 H_{r+1} + \frac{\Lambda_4 H_{r+1}^2}{2} + \frac{\Lambda_5 H_{r+1}^3}{3} + \frac{\Lambda_6 H_{r+1}^4}{4}\right)\left[\alpha'_z|_r - \alpha'_z|_{r+1}\right] + \Gamma_2\left(\Gamma_3 H_{r+1} + \frac{\Gamma_4 H_{r+1}^2}{2} + \frac{\Gamma_5 H_{r+1}^3}{3} + \frac{\Gamma_6 H_{r+1}^4}{4}\right)\left[\beta'_z|_r - \beta'_z|_{r+1}\right]\right\}$$

For k = M → 1 , equation (2.4.41) applies:

$$g_k(x) = b\sum_{k=M+1}^{k+1}\left\{\bar{g}\,(x, H_r)\left[\alpha'_z|_r - \alpha'_z|_{r-1}\right] + \bar{h}\,(x, H_r)\left[\beta'_z|_r - \beta'_z|_{r-1}\right]\right\} \quad (2.4.41)$$

or

$$g_k(x) = g_k = b\sum_{r=M+1}^{k+1}\left\{\Lambda_2\left(\Lambda_3 H_r + \Lambda_4\frac{H_r^2}{2} + \Lambda_5\frac{H_r^3}{3} + \Lambda_6\frac{H_r^4}{4}\right)\left(\alpha'_z|_r - \alpha'_z|_{r-1}\right) + \Gamma_2\left(\Gamma_3 H_r + \Gamma_4\frac{H_r^2}{2} + \Gamma_5\frac{H_r^3}{3} + \Gamma_6\frac{H_r^4}{4}\right)\left(\beta'_z|_r - \beta'_z|_{r-1}\right)\right\}$$

Also, in accordance with equations (2.4.18) and (2.4.22)

$$g_{M+1} = 0$$

Now examine the form for $\bar{\ell}_k(x,z)$, equation (2.4.48), for $k = 1 \rightarrow 2M + 1$

$$\bar{\ell}_k(x,z) = -\alpha'_z\big|_k \int_0^z \bar{g}(x,z)\,dz - \beta'_z\big|_k \int_0^z \bar{h}(x,z)\,dz \qquad (2.4.48)$$

or

$$\bar{\ell}_k(z) = -\alpha'_z\big|_k \Lambda_2\left(\frac{\Lambda_3 z^2}{2} + \frac{\Lambda_4 z^3}{6} + \frac{\Lambda_5 z^4}{12} + \frac{\Lambda_6 z^5}{20}\right) - \beta'_z\big|_k \Gamma_2\left(\frac{\Gamma_3 z^2}{2} + \frac{\Gamma_4 z^3}{6} + \frac{\Gamma_5 z^4}{12} + \frac{\Gamma_6 z^5}{20}\right)$$

In the solution for the functions of integration $C_k(x)$, we must solve for $\ell_k(x)$:

(a) for $k = M + 2 \rightarrow 2M + 1$

$$\ell_k(x) = \sum_{r=M+1}^{k-1} \Big\{ -H_{r+1}\left(g_r(x) - g_{r+1}(x)\right) + \left(\bar{\ell}_r(x, H_{r+1}) - \bar{\ell}_{r+1}(x, H_{r+1})\right)\Big\} \qquad (2.4.58)$$

or

$$\ell_k(x) = \ell_k = \sum_{r=M+1}^{k-1} \Big\{ -H_{r+1}\left(g_r - g_{r+1}\right) + \left(-\alpha'_z\big|_r + \alpha'_z\big|_{r+1}\right)\Lambda_2\left(\frac{\Lambda_3 H_{r+1}^2}{2} + \frac{\Lambda_4 H_{r+1}^3}{6} + \frac{\Lambda_5 H_{r+1}^4}{12} + \frac{\Lambda_6 H_{r+1}^5}{20}\right) + \left(-\beta'_z\big|_r + \beta'_z\big|_{r+1}\right)\Gamma_2\left(\frac{\Gamma_3 H_{r+1}^2}{2} + \frac{\Gamma_4 H_{r+1}^3}{6} + \frac{\Gamma_5 H_{r+1}^4}{12} + \frac{\Gamma_6 H_{r+1}^5}{20}\right)\Big\}$$

(b) for $k = M + 1$

$$\ell_K(x) = \sum_{r=M+1}^{k+1} \Big\{ -H_r \big(g_{r-1}(x) - g_r(x) \big) + \big(\bar{\ell}_{r-1}(x, H_r) - \bar{\ell}_r(x, H_r) \big) \Big\} \quad (2.4.66)$$

or

$$\ell_k(x) = \ell_k = \sum_{r=M+1}^{k+1} \Big\{ -H_r \big(g_{r-1} - g_r \big) + \big(-\alpha'_z\big|_{r-1} + \alpha'_z\big|_r \big) \Lambda_2 \left(\frac{\Lambda_3 H_r^2}{2} + \frac{\Lambda_4 H_r^3}{6} + \frac{\Lambda_5 H_r^4}{12} + \frac{\Lambda_6 H_r^5}{20} \right) + \big(-\beta'_z\big|_{r-1} + \beta'_z\big|_r \big) \Gamma_2 \left(\frac{\Gamma_3 H_r^2}{2} + \frac{\Gamma_4 H_r^3}{6} + \frac{\Gamma_5 H_r^4}{12} + \frac{\Gamma_6 H_r^5}{20} \right) \Big\}$$

The next step combines a number of the above *functions for convenience:*

$$-g_K(x)\, z + \bar{\ell}_K(x, z) + \ell_K(x) \equiv \bar{\bar{h}}_K(x, z) \quad (2.4.68)$$

or

$$-z\, g_K + \bar{\ell}_K(z) + \ell_K = \bar{\bar{h}}_K(z)$$

Since, for these temperature and moisture profiles, $\frac{\partial \bar{\bar{h}}}{\partial x} k = 0$, we have no $\bar{h}$ term in the M equation (2.5.6) and no h* term in the N equation (2.5.18). We do, however, have M^T, M^m, N^T, and N^m terms shown as follows.

For M^T , use equation (2.5.15), insert the ΔT form (2.7.1), and note that the term $(\Lambda_1 + \Lambda_2 x)$ may be pulled through the summation sign.

$$M^T(x) = (\Lambda_1 + \Lambda_2 x)\sum_{k=1}^{2M+1} \frac{1}{1 - c\bar{Q}_{13}\bar{S}'_{13}|_k} \Big\{ \Big[\bar{Q}_{11}\alpha_x|_k + \bar{Q}_{12}\alpha_y|_k - c\bar{Q}_{13}|_k(\alpha'_z - \alpha_z)_k + \bar{Q}_{16}\alpha_{xy}|_k \Big] \cdot \Big[\Lambda_3\frac{(H_{k+1}^2 - H_k^2)}{2} + \Lambda_4\frac{(H_{k+1}^3 - H_k^3)}{3} + \Lambda_5\frac{(H_{k+1}^4 - H_k^4)}{4} + \Lambda_6\frac{(H_{k+1}^5 - H_k^5)}{5} \Big] \Big\}$$

Similarly,

$$M^m(x) = (\Gamma_1 + \Gamma_2 x)\sum_{k=1}^{2M+1} \frac{1}{1 - c\bar{Q}_{13}\bar{S}'_{13}|_k} \Big\{ \Big[\bar{Q}_{11}\beta_x|_k + \bar{Q}_{12}\beta_y|_k - c\bar{Q}_{13}|_k(\beta'_z - \beta_z)_k + \bar{Q}_{16}\beta_{xy}|_k \Big] \cdot \Big[\Gamma_3\frac{(H_{k+1}^2 - H_k^2)}{2} + \Gamma_4\frac{(H_{k+1}^3 - H_k^3)}{3} + \Gamma_5\frac{(H_{k+1}^4 - H_k^4)}{4} + \Gamma_6\frac{(H_{k+1}^5 - H_k^5)}{5} \Big] \Big\}$$

Now, refer to the N definition (2.5.27) and again insert the ΔT form (2.7.1),

$$N^T = (\Delta_1 + \Delta_2 x) \sum_{k=1}^{2M+1} \frac{1}{1 - c\bar{Q}_{13} S'_{13}|_k} \Big\{ \Big[\bar{Q}_{11} \alpha_x|_k + \bar{Q}_{12} \alpha_y|_k - c\bar{Q}_{13}|_k (\alpha'_z - \alpha_z)_k + \bar{Q}_{16} \alpha_{xy}|_k \Big] \cdot \Big[\Delta_3 (H_{k+1} - H_k) + \Delta_4 \frac{(H_{k+1}^2 - H_k^2)}{2} + \Delta_5 \frac{(H_{k+1}^3 - H_k^3)}{3} + \Delta_6 \frac{(H_{k+1}^4 - H_k^4)}{4} \Big] \Big\}$$

Similarly,

$$N^m = (\Gamma_1 + \Gamma_2 x) \sum_{k=1}^{2M+1} \frac{1}{1 - c\bar{Q}_{13} S'_{13}|_k} \Big\{ \Big[\bar{Q}_{11} \beta_x|_k + \bar{Q}_{12} \beta_y|_k - c\bar{Q}_{13}|_k (\beta'_z - \beta_z)_k + \bar{Q}_{16} \beta_{xy}|_k \Big] \cdot \Big[\Gamma_3 (H_{k+1} - H_k) + \Gamma_4 \frac{(H_{k+1}^2 - H_k^2)}{2} + \Gamma_5 \frac{(H_{k+1}^3 - H_k^3)}{3} + \Gamma_6 \frac{(H_{k+1}^4 - H_k^4)}{4} \Big] \Big\}$$

2.8 Summary of Formulae for Stress; Calculation of Strains

Once the basic variables have been obtained, (see Section 3.1), there is one most direct order in which to solve for the stresses. The following list should assist the reader.

σ_x	From (2.5.2)
σ_z	From (2.3.11) with (2.3.15), (2.3.14), (2.3.13)
τ_{xz}	From (2.3.1) with (2.3.2), (2.3.3), (2.3.4), and (2.3.8)
τ_{yz}	From τ_{xz}, with (2.4.1b)
τ_{xy}	From σ_x and σ_z, with (2.4.10)
σ_y	From σ_x, σ_z, τ_{xy} with (2.4.8)

Once these stresses have been computed, we may calculate the strains by use of the matrix equation (2.1.17) in the following form:

$$\begin{bmatrix} \varepsilon_x \\ 0 \\ \varepsilon_z \\ 0 \\ \varepsilon_{xz} \\ 0 \end{bmatrix}_k = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{13} & 0 & 0 & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{23} & 0 & 0 & \bar{S}_{26} \\ \bar{S}_{13} & \bar{S}_{23} & \bar{S}_{33} & 0 & 0 & \bar{S}_{36} \\ 0 & 0 & 0 & \bar{S}_{44} & \bar{S}_{45} & 0 \\ 0 & 0 & 0 & \bar{S}_{45} & \bar{S}_{55} & 0 \\ \bar{S}_{61} & \bar{S}_{62} & \bar{S}_{63} & 0 & 0 & \bar{S}_{66} \end{bmatrix}_k \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix}_k + \begin{bmatrix} \alpha_x \Delta T + \beta_x m \\ \alpha_y \Delta T + \beta_y m \\ \alpha_z \Delta T + \beta_z m \\ 0 \\ 0 \\ \alpha_{xy} \Delta T + \beta_{xy} m \end{bmatrix}_k$$

CHAPTER 3

Studies of a Single Plate Element

3.1 Solution for a Single Plate Element Subject to Known Surface Stresses

We examine all of the equations and unknown functions present at this step. The temperature and moisture effects are the result of known functions. Assume that all surface normal and shear stresses are known. There are:

Eight known functions: τ_L, τ_u, σ_L, σ_u, N^T, N^m, M^T, M^m ;

Six Unknown functions: $u°$, $w°$, ϕ, N, M, Q ;

Equations: 3 equilibrium equations (2.6.1, 2.6.2, 2.6.3)
3 definitions of M,N,Q (2.5.6, 2.5.18, 2.5.31)
6 total

Thus, we seem to have exactly as many unknowns as equations. However, we also have the equation:

$$\sigma_L = \sigma_u + \left[\frac{2h}{3} \bar{Q}_{55}\big|_1 - \bar{\alpha}_{2M+1} \right] \frac{d\phi}{dx} \qquad (2.3.16)$$

If we differentiate the stress resultant equation (2.5.31) and solve for $\frac{dQ}{dx}$, we get

$$\frac{dQ}{dx} = k_5 \frac{d\phi}{dx}$$

and compare this with equilibrium equation (2.6.2),

$$\frac{dQ}{dx} = \sigma_L - \sigma_u$$

We can combine these equations to get

$$\sigma_L - \sigma_u = k_5 \frac{d\phi}{dx} \tag{3.1.1}$$

A quick comparison between (2.3.16) and (3.1.1) shows that we must have

$$k_5 = \frac{2h}{3} Q_{55}\Big|_1 - \bar{\alpha}_{2M+1} \tag{3.1.2}$$

or else we have an additional equation which is inconsistent. By a check of examples, including a five-layer laminate, and by use of the symmetry of b_{55} and material properties and the anti-symmetry of H_k , equation (3.1.2) has been verified. Thus, we do have exactly as many equations (six) as unknowns.

We will now proceed to systematically eliminate variables in order to arrive at equations for the displacement variables $u°$ and $w°$. Substitute the integrated quantities (2.5.6), (2.5.18), and (2.5.31) into the equilibrium equations (2.6.1), (2.6.2) and (2.6.3):

$$0 = A\frac{d^4\phi}{dx^4} + B\frac{d^2\phi}{dx^2} + C\frac{d^4\tau_L}{dx} + D\frac{d^2\tau_L}{dx^2} + \overline{C}\frac{d^4\tau_u}{dx^2}$$
$$+ \overline{D}\frac{d^2\tau_u}{dx^2} + E\frac{d^3\sigma_L}{dx^3} + \overline{A}\frac{d^2u^0}{dx^2} - \frac{dN^T}{dx} - \frac{dN^m}{dx}$$
$$+ \overline{E}\frac{d\sigma_L}{dx} + \frac{dh^*}{dx} - \tau_L + \tau_u \qquad (3.1.3)$$

$$0 = k_5\frac{d\phi}{dx} - \sigma_L + \sigma_u \qquad (3.1.4)$$

$$0 = -D_{11}\frac{d^3w^0}{dx^3} + F\frac{d^4\phi}{dx^4} + G\frac{d^2\phi}{dx^2} + H\frac{d^4\tau_L}{dx^4}$$
$$+ I\frac{d^2\tau_L}{dx^2} + \overline{H}\frac{d^4\tau_u}{dx^4} + \overline{I}\frac{d^2\tau_u}{dx^2} + \frac{d\overline{h}}{dx}$$
$$- \frac{dM^T}{dx} - \frac{dM^m}{dx} - k_5\phi + \frac{h}{2}(\tau_L + \tau_u) \qquad (3.1.5)$$

Solve equation (3.1.4) for $\frac{d\phi}{dx}$. It may be directly substituted into (3.1.3), and substituted into (3.1.5) after (3.1.5) has been differentiated once.

$$\frac{d\phi}{dx} = \frac{1}{k_5}(\sigma_L - \sigma_u) \qquad (3.1.4b)$$

$$0 = \frac{A}{k_5}\left[\frac{d^3\sigma_L}{dx^3} - \frac{d^3\sigma_u}{dx^3}\right] + \frac{B}{k_5}\left[\frac{d\sigma_L}{dx} - \frac{d\sigma_u}{dx}\right] + C\frac{d^4\tau_L}{dx^4}$$
$$+ D\frac{d^2\tau_L}{dx^2} + \overline{C}\frac{d^4\tau_u}{dx^4} + \overline{D}\frac{d^2\tau_u}{dx^2} + E\frac{d^3\sigma_L}{dx^3}$$
$$+ \overline{A}\frac{d^2u^0}{dx^2} - \frac{dN^T}{dx} - \frac{dN^m}{dx} + \overline{E}\frac{d\sigma_L}{dx} + \frac{dh}{dx} - \tau_L + \tau_u \qquad (3.1.6a)$$

or, using an operator notation where $D_x = \frac{d}{dx}$

$$\frac{d^2u^\circ}{dx^2} = -\left[\left(\frac{A}{k_s\bar{A}} + \frac{E}{\bar{A}}\right) D_x^3 + \left(\frac{B}{k_s\bar{A}} + \frac{\bar{E}}{\bar{A}}\right) D_x\right] \sigma_L$$
$$+\left[\left(\frac{A}{k_s\bar{A}}\right) D_x^3 + \left(\frac{B}{k_s\bar{A}}\right) D_x\right] \sigma_u$$
$$-\left[\left(\frac{C}{\bar{A}}\right) D_x^4 + \left(\frac{D}{\bar{A}}\right) D_x^2 - \frac{1}{\bar{A}}\right] \tau_L$$
$$-\left[\left(\frac{\bar{C}}{\bar{A}}\right) D_x^4 + \left(\frac{\bar{D}}{\bar{A}}\right) D_x^2 + \frac{1}{\bar{A}}\right] \tau_u$$
$$+\frac{1}{\bar{A}}\left[\frac{dN^T}{dx} + \frac{dN^m}{dx} - \frac{dh^*}{dx}\right] \qquad (3.1.6b)$$

Now proceed to equation (3.1.5), differentiating once and then substituting (3.1.4b) for $\frac{d\phi}{dx}$

$$0 = -D_{11}\frac{d^4w^\circ}{dx^4} + \frac{F}{k_s}\left[\frac{d^4\sigma_L}{dx^4} - \frac{d^4\sigma_u}{dx^4}\right]$$
$$+\frac{G}{k_s}\left[\frac{d^2\sigma_L}{dx^2} - \frac{d^2\sigma_u}{dx^2}\right] + H\frac{d^5\tau_L}{dx^5} + I\frac{d^3\tau_L}{dx^3}$$
$$+\bar{H}\frac{d^5\tau_u}{dx^5} + \bar{I}\frac{d^3\tau_u}{dx^3} + \frac{d^2\bar{h}}{dx^2} - \frac{d^2M^T}{dx^2}$$
$$-\frac{d^2M^m}{dx^2} - \sigma_L + \sigma_u + \frac{h}{2}\left(\frac{d\tau_L}{dx} + \frac{d\tau_u}{dx}\right) \qquad (3.1.7a)$$

or, again using the operator notation $D_x = \frac{d}{dx}$,

$$\frac{d^4 w^\circ}{dx^4} = \left[\left(\frac{F}{k_5 D_{11}}\right) D_x^4 + \left(\frac{G}{k_5 D_{11}}\right) D_x^2 - \frac{1}{D_{11}} \right] \sigma_L$$

$$+ \left[\left(\frac{-F}{k_5 D_{11}}\right) D_x^4 + \left(\frac{-G}{k_5 D_{11}}\right) D_x^2 + \frac{1}{D_{11}} \right] \sigma_u$$

$$+ \left[\left(\frac{H}{D_{11}}\right) D_x^5 + \left(\frac{-hG}{2k_5 D_{11}} + \frac{I}{D_{11}}\right) D_x^3 + \left(\frac{h}{2D_{11}}\right) D_x \right] \tau_L$$

$$+ \left[\left(\frac{\bar{H}}{D_{11}}\right) D_x^5 + \left(\frac{-hG}{2k_5 D_{11}} + \frac{\bar{I}}{D_{11}}\right) D_x^3 + \left(\frac{h}{2D_{11}}\right) D_x \right] \tau_u$$

$$+ \frac{1}{D_{11}} \left[\frac{d^2 \bar{h}}{dx^2} - \frac{d^2 M^T}{dx^2} - \frac{d^2 M^m}{dx^2} \right] \qquad (3.1.7b)$$

Thus, with the right hand sides of both (3.1.6b) and (3.1.7b) known, the solutions in terms of displacements are given by

$$u^\circ(x) = A_0 + A_1 x + \int^x \int^x \text{right hand side of (3.1.6b)} \qquad (3.1.8)$$

$$w^\circ(x) = B_0 + B_1 x + B_2 x^2 + B_3 x^3 + \int^x \int^x \int^x \int^x \text{right hand side of (3.1.7b)} \qquad (3.1.9)$$

where $\int^x$ indicate an indefinite integral.

The evaluation of these constants in terms of real boundary condition will not only involve w^0 and u^0 , but also Q M, and N. It is necessary, therefore, to derive expressions for M, Q, and N in terms of u° and w°.

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AN ANALYTICAL MODEL FOR BONDED JOINT ANALYSIS IN COMPOSITE STRU--ETC(U)
FEB 77 R C WETHERHOLD , J R VINSON AFOSR-74-2739
UNCLASSIFIED MAE-TR-198 AFOSR-TR-78-1337 NL

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Return to equation (2.5.6) and substituting in (3.1.4b) for $\frac{d\phi}{dx}$,

$$M = -D_{11}\frac{d^2w^0}{dx^2} + \frac{F}{k_5}\left[\frac{d^2\nabla_L}{dx^2} - \frac{d^2\nabla_u}{dx^2}\right] + \frac{G}{k_5}(\nabla_L - \nabla_u) + H\frac{d^3\tau_L}{dx^3} + I\frac{d\tau_L}{dx} + \bar{H}\frac{d^3\tau_u}{dx^3} + \bar{I}\frac{d\tau_u}{dx} + \bar{h}_{(x)} - M^T - M^m \qquad (3.1.10a)$$

or, with the operator notation $D_x = \frac{d}{dx}$,

$$M = -D_{11}\frac{d^2w^0}{dx^2} + \left[\left(\frac{F}{k_5}\right)D_x^2 + \left(\frac{G}{k_5}\right)\right]\nabla_L + \left[\left(\frac{-F}{k_5}\right)D_x^2 - \left(\frac{G}{k_5}\right)\right]\nabla_u + \left[(H)D_x^3 + (I)D_x\right]\tau_L + \left[(\bar{H})D_x^3 + (\bar{I})D_x\right]\tau_u + \left[\bar{h} - M^T - M^m\right] \qquad (3.1.10b)$$

A similar substitution may be performed with equations (2.5.18) and (3.1.4b)

$$N = \frac{A}{k_5}\left[\frac{d^2\nabla_L}{dx^2} - \frac{d^2\nabla_u}{dx^2}\right] + \frac{B}{k_5}(\nabla_L - \nabla_u) + C\frac{d^3\tau_L}{dx^3} + D\frac{d\tau_L}{dx} + \bar{C}\frac{d^3\tau_u}{dx^3} + \bar{D}\frac{d\tau_u}{dx} + E\frac{d^2\nabla_L}{dx^2} + \bar{A}\frac{du^0}{dx} - N^T - N^m + h^*_{(x)} + \bar{E}\nabla_L \qquad (3.1.11a)$$

or,

$$N = \bar{A}\frac{du^\circ}{dx} + \left[\left(\frac{A}{k_s}+E\right)D_x^3 + \left(\frac{B}{k_s}+\bar{E}\right)\right]\sigma_L$$
$$+\left[\left(\frac{-A}{k_s}\right)D_x^2 - \frac{B}{k_s}\right]\sigma_u + \left[(C)D_x^3 - (D)D_x\right]\tau_L$$
$$+\left[(\bar{C})D_x^3 + (\bar{D})D_x\right]\tau_u + \left[-N^T - N^m + h^*_{(x)}\right] \qquad (3.1.11b)$$

To obtain a relation between Q and w^0, substitute (3.1.10a) into (2.6.3):

$$-D_{11}\frac{d^3w^\circ}{dx^3} + \frac{F}{k_s}\left[\frac{d^3\sigma_L}{dx^3} - \frac{d^3\sigma_u}{dx^3}\right] + \frac{G}{k_s}\left(\frac{d\sigma_L}{dx} - \frac{d\sigma_u}{dx}\right)$$
$$+ H\frac{d^4\tau_L}{dx^4} + I\frac{d^2\tau_L}{dx^2} + \bar{H}\frac{d^4\tau_u}{dx^4} + \bar{I}\frac{d^2\tau_u}{dx^4} + \frac{d\bar{h}}{dx}$$
$$-\frac{dM^T}{dx} - \frac{dM^m}{dx} - Q + \frac{h}{2}(\tau_L+\tau_u) = 0 \qquad (3.1.12a)$$

or, using the operator notation,

$$Q = -D_{11}\frac{d^3w^\circ}{dx^2} + \left[\frac{F}{k_s}D_x^3 + \frac{G}{k_s}D_x\right]\sigma_L$$
$$+\left[-\frac{F}{k_s}D_x^3 - \frac{G}{k_s}D_x\right]\sigma_u + \left[H D_x^4 + I D_x^2 + \frac{h}{2}\right]\tau_L$$
$$+\left[\bar{H}D_x^4 + \bar{I}D_x^2 + \frac{h}{2}\right]\tau_u + \left[\frac{d\bar{h}}{dx} - \frac{dM^T}{dx} - \frac{dM^m}{dx}\right] \qquad (3.1.12b)$$

Thus, if we use (3.1.11b), and (3.1.12b), (3.1.10b) to solve for the boundary value constants in equations (3.1.8) and (3.1.9), the other variables can be computed as follows.

N given by (3.1.11b)

M given by (3.1.10b)

Q given by (3.1.12b), or, more simply, by integrating (2.6.2)

and, rearranging (2.5.31),

$$\phi = \frac{1}{k_5} Q \tag{2.5.31b}$$

3.2 Parametric Studies of a Single Plate Element Subjected to Known Surface Tractions and Known Temperature and Moisture Profiles

In the interest of demonstrating the practicality and consistency of this method, several typical examples have been selected. These examples include a plate subjected to simple normal tension, simple shear stress, and various temperature and moisture profiles. Where a specific example is needed, material properties for a laminate of T300/5208 graphite epoxy are used. (See Fig. 8) Figures 9 and 10 will help visualization of the boundary conditions. The heights and laminae are numbered as shown in Fig. 5; $h_0 = .1397 \times 10^{-3}$m is the thickness of each lamina.

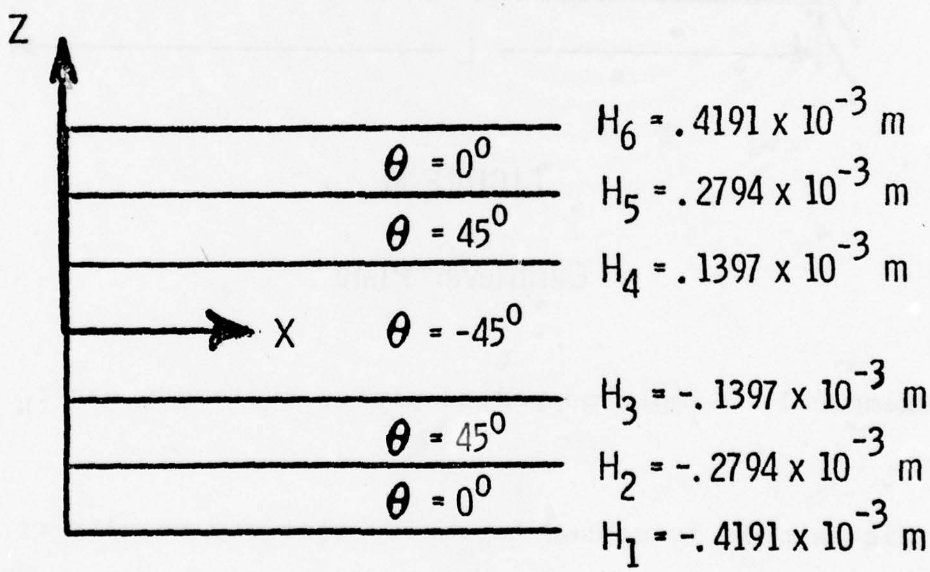

FIGURE 8

Example Symmetric Laminate

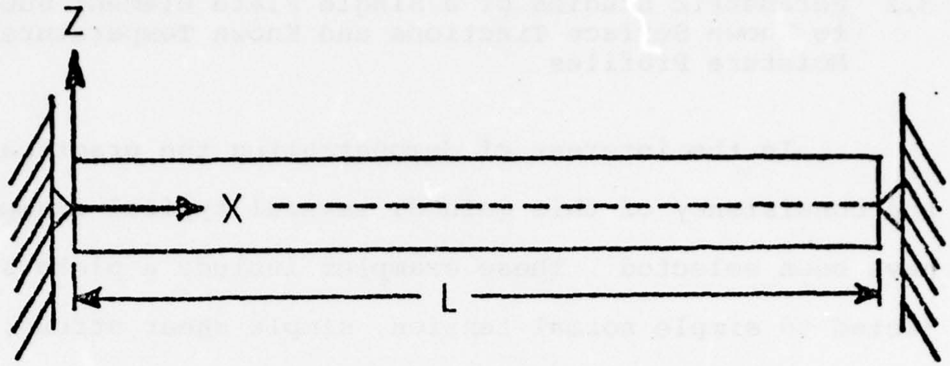

FIGURE 9

Simply Supported Plate

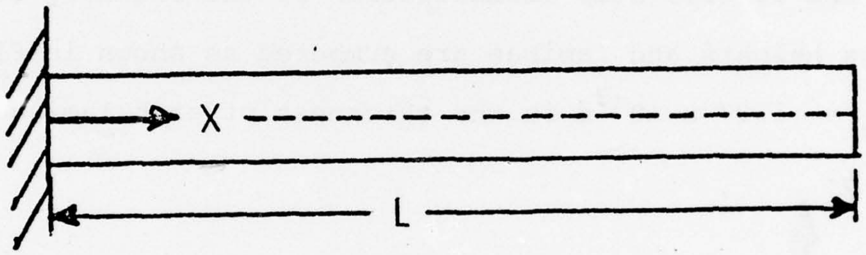

FIGURE 10

Cantilever Plate

Example 1: Simply supported plate, surface shear $\tau_L = \tau_u = \tau$.

This example is chosen to verify that the results of this method are consistent with physical intuition. The displacement $u = u(z)$ and the shear stress τ_{xz} will be calculated.

The boundary conditions are:

$$u^0(0) = u^0(L) = 0$$
$$w^0(0) = w^0(L) = M(0) = M(L) = 0$$

By 3.1.9 and 3.1.10b,

$$w^0(x) \equiv 0$$

By 3.1.8, with $\tau_u = \tau_L = \tau$

$$u^0(x) \equiv 0$$

By 2.4.67, with $\Delta T = m = 0$

$$\begin{aligned} u(x,z) = u(z) = &\Big[\bar{S}'_{13}\big|_k \left(-\frac{z^2}{2h} + \frac{z^3}{h^2}\right) + (\Delta + \bar{S}_{55})_k \left(-\frac{z}{2} - \frac{z^2}{h} + \frac{2z^3}{h^2}\right) \\ &+ G_k - \delta_k z \Big] \tau \\ &+ \Big[\bar{S}'_{13}\big|_k \left(\frac{z^2}{2h} + \frac{z^3}{h^2}\right) + (\Delta + \bar{S}_{55})_k \left(\frac{-z}{2} + \frac{z^2}{h} + \frac{2z^3}{h^2}\right) \\ &+ \bar{G}_k - \bar{\delta}_k z \Big] \tau \\ &+ \Big[(2 + 2\Delta \bar{Q}_{55})_k \left(z - \frac{4z^3}{3h^2}\right) + (2\Delta + 2\bar{S}_{55})_k b_{55}\big|_k z \\ &- \frac{4 \bar{S}'_{13} \bar{Q}_{55}\big|_k}{3h^2} z^3 + E_k - \mu_k z \Big] \phi \end{aligned}$$

where, by 2.5.31b and 3.1.12b,

$$\phi = \frac{1}{k_s} Q = \frac{1}{k_s}\left(\frac{h}{2}\tau + \frac{h}{2}\tau\right) = \frac{h}{k_s}\tau$$

For an all 0^0 lay-up of identical laminae,

$$\bar{S}'_{13}\big|_k = \bar{S}'_{13}$$
$$\bar{Q}_{55}\big|_k = \bar{Q}_{55}$$
$$\Delta_k = G_k = S_k = \bar{G}_k = \bar{S}_k = b_{55}\big|_k = E_k = \mu_k = 0$$

$$u(z) = \left[2\bar{S}'_{13}\frac{z^3}{h^2} + 2\bar{S}_{55}\left(-\frac{z}{2} + \frac{2z^3}{h^2}\right)\right]\tau + \left[2\left(z - \frac{4z^3}{3h^2}\right) - \frac{4\bar{S}'_{13}\bar{Q}_{55}}{3h^2}z^3 \right]\frac{h\tau}{k_s}$$

The form of u(z) is anti-symmetric; u(z) = -u(z). However, it should also be linear in z , which is not obvious. If material properties for T300/5208 are used,

$$E_{11} = 1.434 \times 10^{11} \text{ Pa}$$
$$E_{22} = E_{33} = 1.01 \times 10^{10} \text{ Pa}$$
$$G_{12} = G_{13} = G_{23} = 4.14 \times 10^{9} \text{ Pa}$$
$$\nu_{12} = \nu_{13} = \nu_{23} = .31$$
$$\alpha_1 = 5.13 \times 10^{-6}/°\text{C}$$
$$\alpha_2 = \alpha_3 = 3.05 \times 10^{-5}/°\text{C}$$
$$\beta_1 = 0$$
$$\beta_2 = \beta_3 = 6.67 \times 10^{-3}/\%\text{ moisture}$$

we can plot u(z), as shown in Fig. 11 for an all 0° laminate.

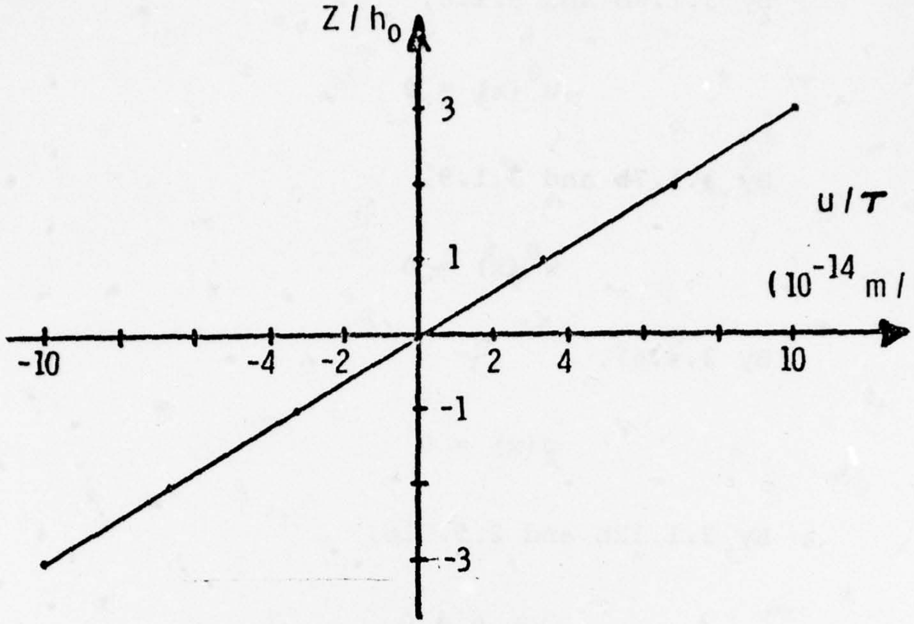

FIGURE 11

Simply Supported Plate, All 0^0 Layers
Plate, Loading: Surface Shear Stress

This practical example shows that the linear terms in z are by far the leading terms; deviation from linearity in u is <1% . With $\tau_{xz} = \bar{Q}_{44}\cancel{\varepsilon_{yz}}^{0} + \bar{Q}_{55}\varepsilon_{xz} = \bar{Q}_{55}\,\frac{1}{2}\,(\frac{\partial u}{\partial z} + \cancel{\frac{\partial w}{\partial x}}^{0})$, τ_{xz} is virtually constant, as expected.

Example 2: *Simply supported plate, normal tension* $\sigma_u = \sigma_L = \sigma$. This example is chosen to check the results of this method with physical intuition. We will compute w=w(z) and also N_x .

$$u^0(0) = u^0(L) = 0$$

$$w^0(0) = w^0(L) = M(0) = M(L) = 0$$

$$\Delta T = m = 0$$

By 3.1.6b and 3.1.8,

$$u^0(x) = 0$$

By 3.1.7b and 3.1.9,

$$w^0(x) = 0$$

By 2.4.67,

$$u(x) = 0$$

By 3.1.12b and 2.5.31b,

$$Q = \phi = 0$$

Therefore, by 2.4.42,

$$\frac{\partial w}{\partial x} = 0 \ ,$$

or

$$w = w(z)$$

For the case of all 0^0 layers of identical material,

$$w(z) = w^0 + \int_o^z \varepsilon_z \, dz$$

$$= \int_o^z \varepsilon_z \, dz$$

By 2.3.11 and 2.3.15

$$\sigma_z \equiv \sigma$$

By 2.5.2b,

$$\sigma_x = c\,\bar{Q}_{13}\bar{S}'_{33}\sigma_L = c\,\bar{Q}_{13}\bar{S}'_{33}\sigma$$

Thus, $N = \int_{-h/2}^{h/2} \sigma_x\, dz = h\sigma_x > 0$, which is as expected; the plate must be in tension to stay in place.

Finally, by 2.4.11c,

$$\varepsilon_z = \bar{S}'_{13}(c\,\bar{Q}_{13}\bar{S}'_{33}\sigma) + \bar{S}'_{33}\sigma$$

$$\varepsilon_z = \bar{S}'_{33}(1+c\,\bar{Q}_{13}\bar{S}'_{13})\sigma$$

Therefore,

$$w(z) = \varepsilon_z\, z = ¢\sigma z\ ; \qquad ¢ > 0$$

as expected. For T300/5208, this relationship, with tracing constant $c = 1$, becomes

$$w(z) = (8.95 \times 10^{-11}/\mathrm{Pa})\ \sigma\ z$$

and for $c = 0$

$$w(z) = (8.83 \times 10^{-11}/\mathrm{Pa})\ \sigma\ z$$

Example 3: Residual Thermal Stresses

For many advanced composites, the stress free temperature is an elevated cure temperature, and a large negative ΔT loading is present at room temperature. For

residual thermal stress calculation, we may visualize the plate as a cantilever (see Fig. 10). Stresses are generated within the plate due to the mismatch in thermal expansion coefficients between layers.

As boundary conditions, use:

$$w^0(0) = \frac{dw^\circ}{dx}(0) = u^0(0) = 0$$

$$M(L) = N(L) = 0$$

$$\Delta T = \text{constant} \; , \quad m \equiv 0$$

$$M^T = M^m = N^m = 0 \; , \quad N^T \neq 0$$

By 3.1.3 and 3.1.11a

$$u^0(x) \equiv \frac{N^T}{\bar{A}} x$$

By 3.1.9, 3.1.12b, and 3.1.10a

$$w^0(x) \equiv 0$$

By 3.1.12b and 2.5.31b,

$$\phi = Q = 0$$

By 2.5.2b,

$$\sigma_x(x,z)_k = \sigma_x\big|_k = \frac{1}{1 - c\bar{Q}_{13}\bar{S}'_{13}\big|_k}\Big[\bar{Q}_{11}\big|_k \frac{du^\circ}{dx} - \big(\bar{Q}_{11}\alpha_x\big|_k + \bar{Q}_{12}\alpha_y\big|_k + c\,\bar{Q}_{13}\big|_k(\alpha_z - \alpha'_z)_k + \bar{Q}_{16}\alpha_{xy}\big|_k\big)\Delta T\Big]$$

If all layers are 0^0, then

$$\sigma_x = \tau_{xy} = 0 \quad \text{as expected}$$

For the $[0, \pm 45]_s$ laminate shown in Fig. 8, the stresses are as shown in Fig. 12.

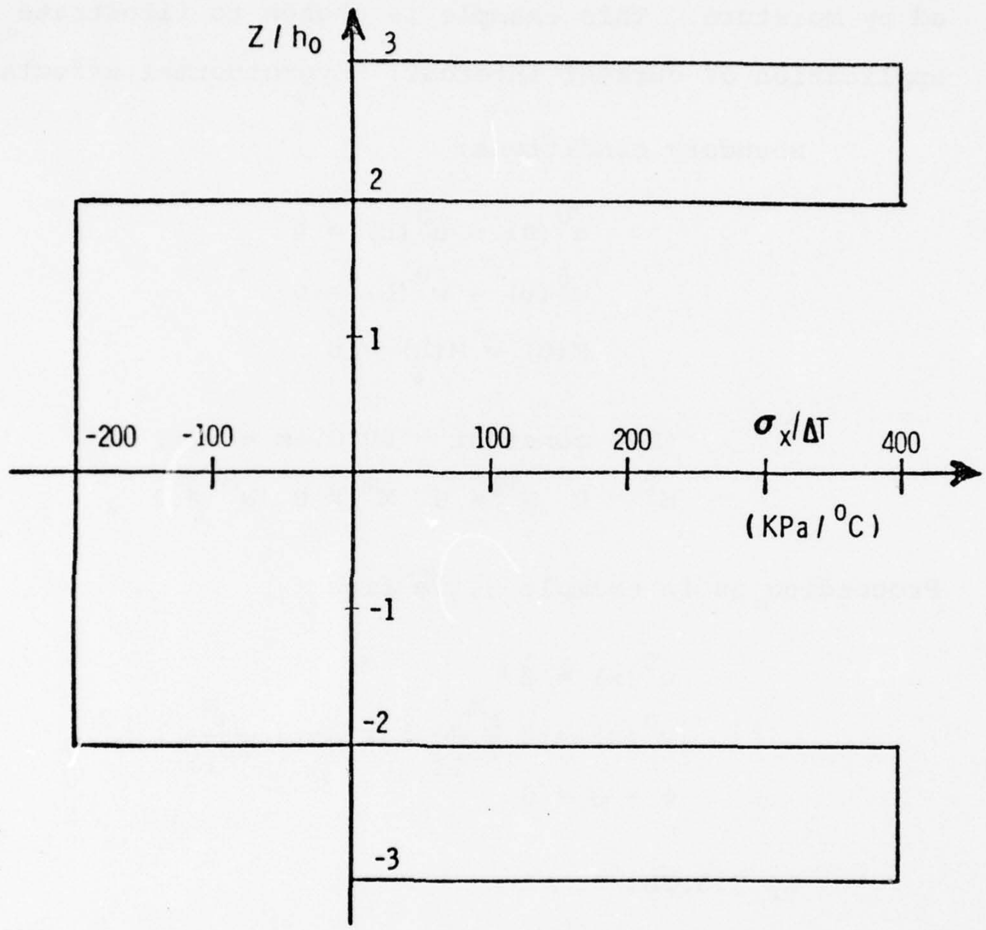

FIGURE 12

Residual Thermal Stresses

For cooldown to room temperature, $\Delta T < 0$, the 0^0 layers would be in compression and the ±45° layers in tension.

Example 4: Structural plate element subjected to temperature increase and moisture infiltration.

For this case, the plate is constrained by the supports and then uniformly heated and non-uniformly infiltrated by moisture. This example is chosen to illustrate an application of current interest: hygrothermal effects.

Boundary conditions:

$$u^0(0) = u^0(L) = 0$$
$$w^0(0) = w^0(L) = 0$$
$$M(0) = M(L) = 0$$

$$\Delta T = \text{constant} = 80°C, \quad m = m(z)$$
$$M^T = 0 \quad N^T \neq 0 \quad M^m \neq 0 \quad N^m \neq 0$$

Proceeding as in example 3, we find

$$u^0(x) = 0$$
$$w^0(x) = \left(\frac{M^m L}{2D_{11}}\right) x - \left(\frac{M^m}{2D_{11}}\right) x^2$$
$$\phi = Q = 0$$

By 2.5.2b,

$$\sigma_x(z)_k = \frac{1}{1-c\bar{Q}_{13}\bar{S}'_{13}|_k}\left[-\bar{Q}_{11}|_k\, z\frac{d^2w^0}{dx^2}\right.$$
$$-\left(\bar{Q}_{11}\alpha_x|_k + \bar{Q}_{12}\alpha_y|_k + c\bar{Q}_{13}(\alpha_z-\alpha'_z)_k + \bar{Q}_{16}\alpha_{xy}|_k\right)\Delta T$$
$$\left.-\left(\bar{Q}_{11}\beta_x|_k + \bar{Q}_{12}\beta_y|_k + c\bar{Q}_{13}(\alpha_z-\alpha'_z)_k + \bar{Q}_{16}\beta_{xy}|_k\right) m\right]$$

The case of one-sided moisture distribution [Ref. 3], with the diffusion process assumed stress independent, is shown in Fig. 13. Three curves have been selected for stress calculations, with $m_0 = 1.4\%$.

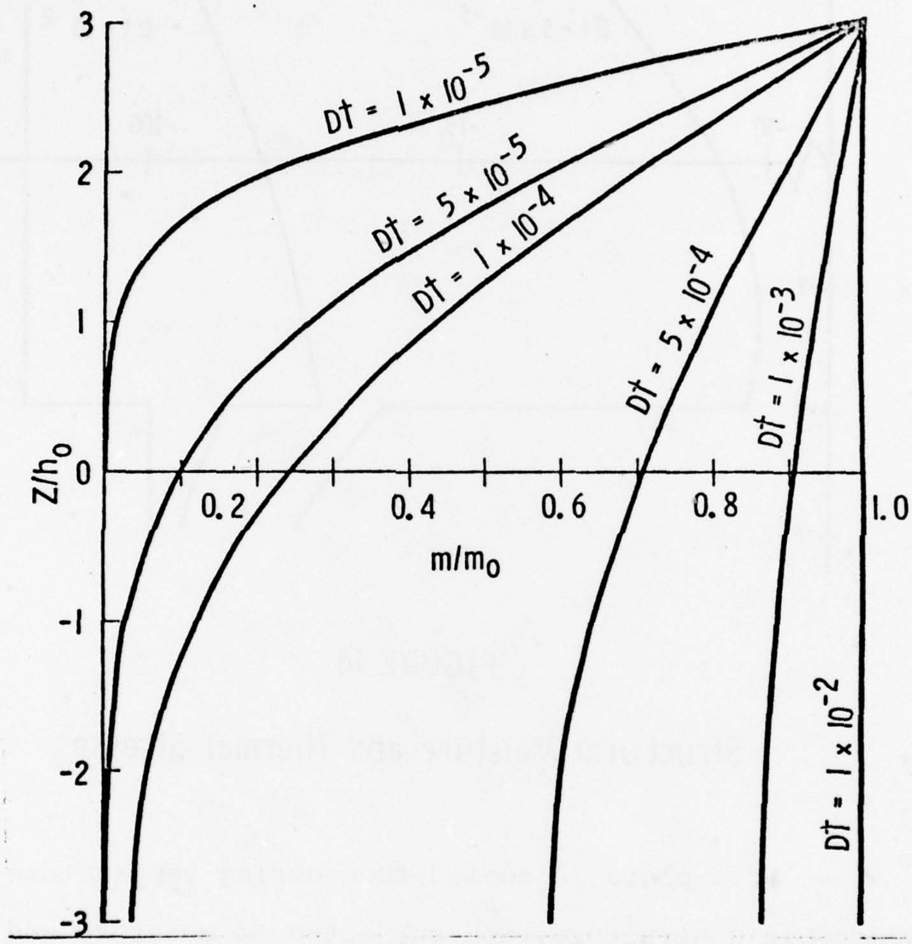

FIGURE 13

Single Side Moisture Diffusion Profile

The resultant stresses are shown in Fig. 14.

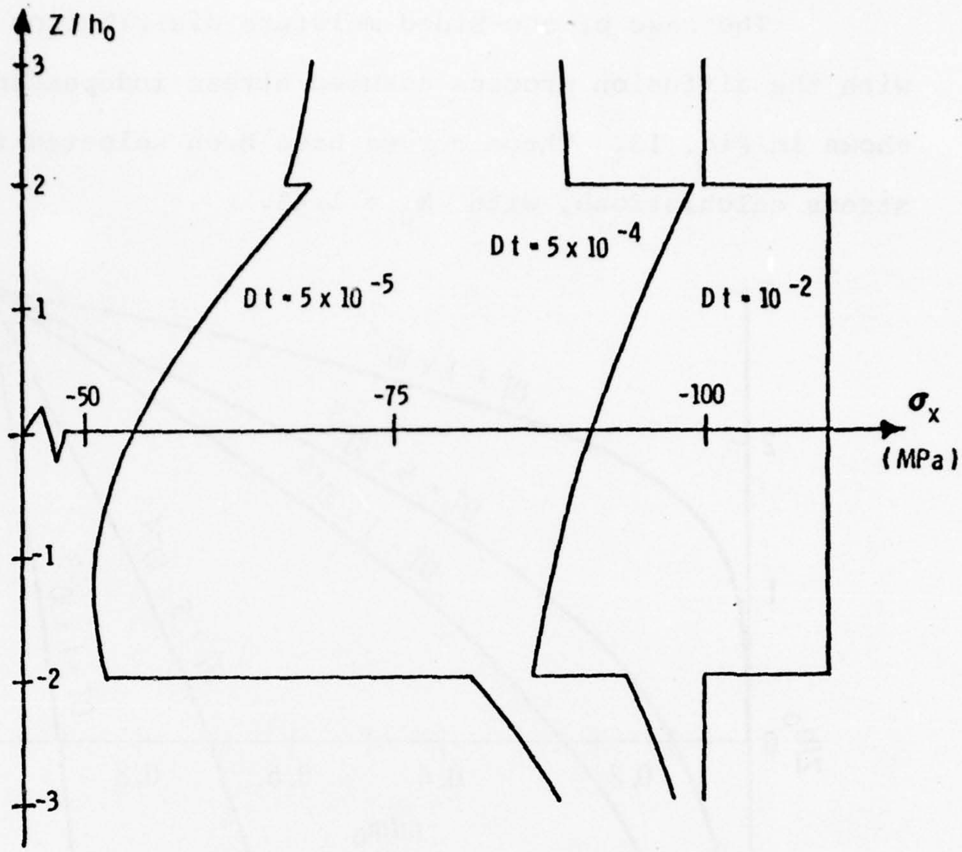

FIGURE 14

Structural Moisture and Thermal Stresses

If a plate is cooled from curing temperature, placed in a structure, and subjected to moisture and elevated temperature, the results can be analyzed by superimposing examples three and four.

For cooldown from a cure temperature of 350°F,

the residual curing stresses are the same order as the structural stress due to moisture infiltration. Any analysis which ignores the moisture effect is thus potentially hazardous.

CHAPTER 4

Derivation of the Governing Equations for a Bonded Joint

4.1 Development of Alternate Expressions Employing σ_u

In preparation for a discussion of bonded elements, we need to develop expressions written in terms σ_u, τ_u, τ_L rather than σ_L, τ_u, τ_L. This may be done by employing (3.1.1)

$$\sigma_L - \sigma_u = k_5 \frac{d\phi}{dx} \tag{3.1.1a}$$

or

$$\sigma_L = \sigma_u + k_5 \frac{d\phi}{dx} \tag{3.1.1b}$$

Examine equation (2.4.67), taking first the pertinent term

$$\left(-\bar{S}'_{33}\big|_k \frac{z^2}{2} + H'_k - \epsilon_k z\right)\frac{d\sigma_L}{dx} = \left(-\bar{S}'_{33} \frac{z^2}{2} + H'_k - \epsilon_k z\right)\frac{d\sigma_u}{dx}$$
$$+ \left(-k_5 \bar{S}'_{33} \frac{z^2}{2} + k_5 H'_k - k_5 \epsilon_k z\right)\frac{d^2\phi}{dx^2}$$

$$u(x,z)_k = u^\circ(x) - z\frac{dw^\circ}{dx} + \bar{\bar{h}}_k(x,z)$$

$$+\left[\bar{S}'_{13}|_k\left(-\frac{z^2}{2h}+\frac{z^3}{h^2}\right)+(\Delta+\bar{S}_{55})_k\left(-\frac{z}{2}-\frac{z^2}{h}+\frac{2z^3}{h^2}\right)+G_k-\delta_k z\right]\tau_L$$

$$+\left[\bar{S}'_{13}|_k\left(\frac{z^2}{2h}+\frac{z^3}{h^2}\right)+(\Delta+\bar{S}_{55})_k\left(-\frac{z}{2}+\frac{z^2}{h}+\frac{2z^3}{h^2}\right)+\bar{G}_k-\bar{\delta}_k z\right]\tau_u$$

$$+\left[\bar{S}'_{33}|_k\left(-\frac{z^3}{24}-\frac{z^4}{24h}+\frac{z^5}{20h^2}+\frac{hz^2}{16}\right)+F_k-\eta_k z\right]\frac{d^2\tau_L}{dx^2}$$

$$+\left[\bar{S}'_{33}|_k\left(-\frac{z^3}{24}+\frac{z^4}{24h}+\frac{z^5}{20h^2}-\frac{hz^2}{16}\right)+\bar{F}_k-\bar{\eta}_k z\right]\frac{d^2\tau_u}{dx^2}$$

$$+\Big[(2+2\Delta\bar{Q}_{55})_k\left(z-\frac{4z^3}{3h^2}\right)+(2\Delta+2\bar{S}_{55})_k\, b_{55}|_k\, z$$

$$-\frac{4\bar{S}'_{13}\bar{Q}_{55}|_k}{3h^2}z^3+E_k-\mu_k z\Big]\phi$$

$$+\Big\{\bar{S}'_{33}|_k\Big[\bar{Q}_{55}|_k\left(\frac{z^3}{6}-\frac{z^5}{15h^2}\right)-\bar{\alpha}_k\frac{z^2}{2}+\bar{Q}_{55}|_1\frac{hz^2}{6}$$

$$+b_{55}|_k\frac{z^3}{6}\Big]+D_k-\rho_k z-k_5\bar{S}'_{33}\frac{z^2}{2}+k_5H'_k-k_5\epsilon_k z\Big\}\frac{d^2\phi}{dx^2}$$

$$+\left[-\bar{S}'_{33}|_k\frac{z^2}{2}+H'_k-\epsilon_k z\right]\frac{d\sigma_u}{dx} \qquad (4.1.1)$$

Similarly, we may take equation (2.4.42) and alter its form by using (3.1.1) as follows:

$$\frac{\partial w(x,z)_k}{\partial x} = \frac{dw^\circ}{dx}+\left[\bar{S}'_{13}|_k\left(\frac{z}{h}-\frac{3z^2}{h^2}\right)+\delta_k\right]\tau_L+\left[\bar{S}'_{13}|_k\left(-\frac{z}{h}-\frac{3z^2}{h^2}\right)+\bar{\delta}_k\right]\tau_u$$

$$+\left\{\bar{S}'_{33}|_k\left(\frac{z^2}{8}+\frac{z^3}{6h}-\frac{z^4}{4h^2}-\frac{zh}{8}\right)+\eta_k\right\}\frac{d^2\tau_L}{dx^2}$$

$$+\left\{\bar{S}'_{33}|_k\left(\frac{z^2}{8}-\frac{z^3}{6h}-\frac{z^4}{4h^2}+\frac{zh}{8}\right)+\bar{\eta}_k\right\}\frac{d^2\tau_u}{dx^2} \qquad \text{(over)}$$

$$+\left[\frac{4\bar{S}_{13}'\bar{Q}_{55}|_k z^2}{h^2}+\mu_k\right]\phi$$

$$+\Big\{-\bar{S}_{33}'|_k\Big[\bar{Q}_{55}|_k\left(\frac{z^2}{2}-\frac{z^4}{3h^2}\right)-\bar{\alpha}_k z+\bar{Q}_{55}|_1\frac{hz}{3}$$

$$+b_{55}|_k\frac{z^2}{2}\Big]+\rho_k+k_5\bar{S}_{33}'|_k z+k_5\epsilon_k\Big\}\frac{d^2\phi}{dx^2}$$

$$+\left[\bar{S}_{33}'|_k z+\epsilon_k\right]\frac{d\sigma_u}{dx}+\left[g_k+\alpha_z'|_k\bar{g}+\beta_z'|_k\bar{h}\right] \qquad (4.1.2)$$

Next, examine equation (2.5.18):

$$N = A\frac{d^3\phi}{dx^3}+B\frac{d\phi}{dx}+C\frac{d^3\tau_L}{dx^3}+D\frac{d\tau_L}{dx}$$

$$+\bar{C}\frac{d^3\tau_u}{dx^3}+\bar{D}\frac{d\tau_u}{dx}+E\left(\frac{d^2\sigma_u}{dx^2}+k_5\frac{d^3\phi}{dx^3}\right)$$

$$+\bar{A}\frac{du^o}{dx}-N^T-N^m+\bar{E}\left(\sigma_u+k_5\frac{d\phi}{dx}\right)+h^*$$

or, letting $A^* = A + E\,k_5$ (4.1.3)

$B^* = B + \bar{E}\,k_5$ (4.1.4)

$$N = A^*\frac{d^3\phi}{dx^3}+B^*\frac{d\phi}{dx}+C\frac{d^3\tau_L}{dx^3}+D\frac{d\tau_L}{dx}$$

$$+\bar{C}\frac{d^3\tau_u}{dx^3}+\bar{D}\frac{d\tau_u}{dx}+E\frac{d^2\sigma_u}{dx^2}$$

$$+\bar{A}\frac{du^o}{dx}-N^T-N^m+\bar{E}\sigma_u+h^* \qquad (4.1.5)$$

Equation (2.5.31) remains unchanged, containing no explicit σ_L. Equation (2.5.6) also remains unchanged.

4.2 Development of Interface Equations for Bonded Joints

The fundamental assumption on interaction between bonded elements is that the adhesive is a thin, homogeneous, isotropic material in a state of simple shear and normal stress. Examine two elements, "i" and "j" , shown in Figs. 15 and 16 with positive stress and displacements.

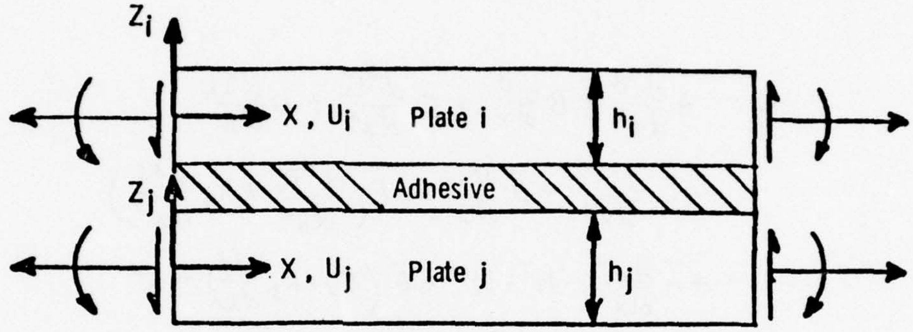

FIGURE 15

Bonded Joint, Adherends "i" and "j"

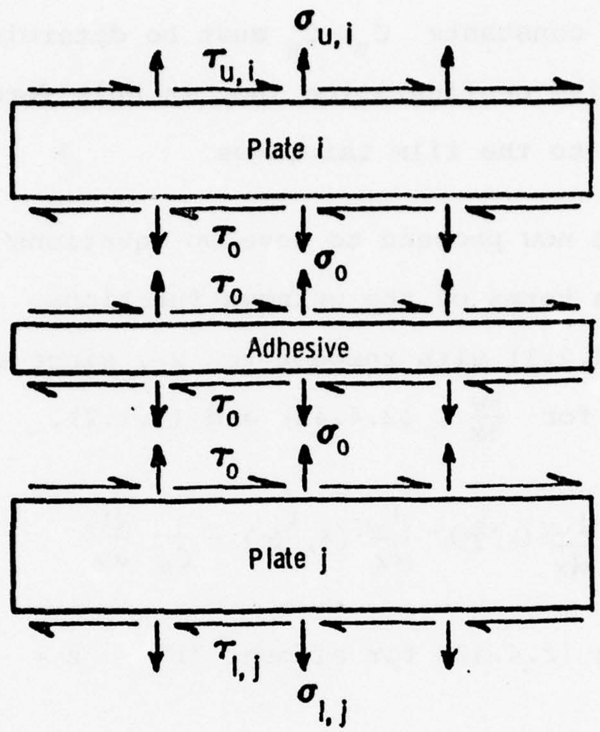

FIGURE 16

Eploded View of Bonded Joint Including Adhesive Stresses

The assumption of simple shear and normal stress may be expressed as:

$$w_i(x, -\frac{h_i}{2}) - w_j(x, \frac{h_j}{2}) = \frac{\sigma_0}{C_N} \qquad (4.2.1)$$

$$u_i\left(x, -\frac{h_i}{2}\right) - u_j\left(x, -\frac{h_j}{2}\right) = \frac{\tau_0}{C_s} \qquad (4.2.2)$$

where the constants C_s, C_N must be determined experimentally in thin film configuration (not in bulk form) and are proportional to the film thickness.

We now proceed to develop equations (4.2.1) and (4.2.2) in terms of the unknown functions. First, differentiate (4.2.1) with respect to x, since we have expressions for $\frac{\partial w}{\partial x}$, (2.4.42) and (4.1.2).

$$\frac{dw_i}{dx}\left(x, -\frac{h_i}{2}\right) - \frac{dw_j}{dx}\left(x, \frac{h_j}{2}\right) = \frac{1}{C_N}\frac{d\sigma_0}{dx} \qquad (4.2.3)$$

Evaluating (2.4.42) for element "i" : $z = -\frac{h_i}{2}$, $k = 1$

$$\left.\frac{\partial w_i(x,z)_k}{\partial x}\right|_{\substack{z=-\frac{h_i}{2}\\ k=1}} \equiv \frac{dw_i}{dx}\left(x, -\frac{h_i}{2}\right)$$

$$\begin{aligned}\frac{dw_i(x,-\frac{h_i}{2})}{dx} = \frac{dw_i^{\circ}}{dx} &+ \left[\bar{S}'_{13}\Big|_1\left(\frac{-h_i/2}{h_i} - \frac{3(h_i/2)^2}{h_i^2}\right) + \delta_1\right]\tau_0 \\ &+ \left[\bar{S}'_{13}\Big|_1\left(\frac{+h_i/2}{h_i} - \frac{3(-h_i/2)^2}{h_i^2}\right) + \bar{\delta}_1\right]\tau_{u,i} \\ &+ \left[\bar{S}'_{33}\Big|_k\left(\frac{(-h_i/2)^2}{8} + \frac{(-h_i/2)^3}{6h_i} - \frac{(-h_i/2)^4}{4h_i^2} - \frac{(-h_i/2)h_i}{8}\right)\right. \\ &\left.+\eta_1\right]\frac{d^2\tau_0}{dx^2}\end{aligned}$$

(over)

$$+\left[\bar{S}'_{33}\Big|_1\left(\frac{(-h_i/2)^2}{8}-\frac{(-h_i/2)^3}{6h}-\frac{(-h_i/2)^4}{4h^2}+\frac{h_i(-h_i/2)}{8}\right)+\bar{\eta}_1\right]\frac{d^2\tau_{u,i}}{dx^2}$$

$$+\left[\frac{4\,\bar{S}'_{13}\,\bar{Q}_{55}|_1}{h_i^2}(-h_i/2)^2+\mu_1\right]\phi_i$$

$$+\left\{-\bar{S}'_{33}\Big|_1\left[\bar{Q}_{55}\Big|_1\left(\frac{(-h_i/2)^2}{2}-\frac{(-h_i/2)^4}{3h_i^2}\right)\right.\right.$$

$$\left.-\cancelto{0}{\bar{\alpha}_1}(-h_i/2)+\bar{Q}_{55}\Big|_1\frac{h_i(-h_i/2)}{3}+\cancelto{0}{b_{55}|_1}\frac{(-h_i/2)^2}{2}\right]$$

$$\left.+\rho_1\right\}\frac{d^2\phi_i}{dx^2}$$

$$+\left(\bar{S}'_{33}\Big|_1\left(\frac{-h_i}{2}\right)+\epsilon_1\right)\frac{dV_0}{dx}+g_1(x)$$

$$+\alpha_z|_1\,\bar{g}(x,-h_i/2)+\beta_z|_1\,\bar{h}(x,-h_i/2)$$

or

$$\frac{dw_i}{dx}\left(x,-\frac{h_i}{2}\right)=\frac{dw_i^0}{dx}+\left[-\frac{5}{4}\bar{S}'_{13}\Big|_1+\delta_1\right]\tau_0+\left[-\frac{1}{4}\bar{S}'_{13}\Big|_1+\bar{\delta}_1\right]\tau_{u,i}$$

$$+\left[\frac{11}{192}h_i^2\,\bar{S}'_{33}\Big|_1+\eta_1\right]\frac{d^2\tau_0}{dx^2}+\left[\frac{-5}{192}h_i^2\,\bar{S}'_{33}\Big|_1+\bar{\eta}_1\right]\frac{d^2\tau_{u,i}}{dx^2}$$

$$+\left[\bar{S}'_{13}\,\bar{Q}_{55}\Big|_1+\mu_1\right]\phi_i$$

$$+\left\{-\bar{S}'_{33}\Big|_1\left[\frac{-3}{48}h_i^2\,\bar{Q}_{55}\Big|_1\right]+\rho_1\right\}\frac{d^2\phi_i}{dx^2}$$

$$+\left(\frac{-h_i}{2}\bar{S}'_{33}\Big|_1+\epsilon_1\right)\frac{dV_0}{dx}+g_1(x)$$

$$+\alpha_z|_1\,\bar{g}\left(x,-\frac{h_i}{2}\right)+\beta_z|_1\,\bar{h}\left(x,-\frac{h_i}{2}\right) \qquad (4.2.4)$$

Similarly, we may use expression (4.1.2) for element "j", with $k = 2M + 1$, $z = +h_i/2$. Note that in the term for $\frac{d^2\phi_j}{dx^2}$, we will use symmetry and substitute $\bar{Q}_{55}|_{2M+1}$ for $\bar{Q}_{55}|_1$. This is done to avoid confusion, so that all quantities subscripted "1" denote element "i", and all quantities subscripted "2M+1" denote element "j".

$$\frac{\partial w_j}{\partial x}(x,z)\Big|_{\substack{z=h_j/2\\k=2M+1}} \equiv \frac{dw_j}{dx}\left(x,\frac{h_j}{2}\right)$$

$$\begin{aligned}
\frac{dw_j}{dx}\left(x,\frac{h_j}{2}\right) = \frac{dw_j^{\circ}}{dx} &+ \Bigg[\bar{S}'_{13}\Big|_{2M+1}\left(\frac{h_j/2}{h_j} - \frac{3(h_j/2)^2}{h_j^2}\right) \\
&+ \delta_{2M+1}\Bigg]\tau_{L,j} \\
&+ \left[\bar{S}'_{13}\Big|_{2M+1}\left(\frac{-h_j/2}{h_j} - \frac{3(h_j/2)^2}{h_j^2}\right) + \bar{\delta}_{2M+1}\right]\tau_o \\
&+ \Bigg\{\bar{S}'_{33}\Big|_{2M+1}\left(\frac{(h_j/2)^2}{8} + \frac{(h_j/2)^3}{6h_j} - \frac{(h_j/2)^4}{4h_j^2} - \frac{(h_j/2)h_j}{8}\right) \\
&+ \eta_{2M+1}\Bigg\}\frac{d^2\tau_{L,j}}{dx^2} \\
&+ \Bigg\{\bar{S}'_{33}\Big|_{2M+1}\left(\frac{(h_j/2)^2}{8} - \frac{(h_j/2)^3}{6h} - \frac{(h_j/2)^4}{4h^2} + \frac{(h_j/2)h_j}{8}\right) \\
&+ \bar{\eta}_{2M+1}\Bigg\}\frac{d^2\tau_o}{dx^2} + \left[\frac{4\bar{S}'_{13}\bar{Q}_{55}|_{2M+1}}{h_j^2}\left(\frac{h_j}{2}\right)^2 + \mu_{2M+1}\right]\phi_j \\
&+ \Bigg\{-\bar{S}'_{33}\Big|_{2M+1}\left[\bar{Q}_{55}\Big|_{2M+1}\left(\frac{(h_j/2)^2}{2} - \frac{(h_j/2)^4}{3h_j^2}\right)\right.
\end{aligned}$$

(over)

$$-\bar{\alpha}_{2M+1}\left(\frac{h_j}{2}\right)+\bar{Q}_{55}\Big|_{2M+1}\frac{h_j(h_j/2)}{3}+\cancel{\gamma_{55}}\Big|_{2M+1}\frac{(h_j/2)^2}{2}\Big]$$

$$+\rho_{2M+1}+k_5\,\bar{S}'_{33}\Big|_{2M+1}\left(\frac{h_j}{2}\right)+k_5\,\epsilon_{2M+1}\Big\}\frac{d^2\phi_j}{dx^2}$$

$$+\left[\bar{S}'_{33}\Big|_{2M+1}\left(\frac{h_j}{2}\right)+\epsilon_{2M+1}\right]\frac{dU_0}{dx}$$

$$+\;g_{2M+1}(x)\;+\alpha_z\Big|_{2M+1}\bar{g}\left(x,\frac{h_j}{2}\right)+\beta_z\Big|_{2M+1}\left(x,\frac{h_j}{2}\right)$$

or

$$\frac{dw_j(x,\frac{h_j}{2})}{dx}=\frac{dw_j^0}{dx}+\left[-\frac{1}{4}\,\bar{S}'_{13}\Big|_{2M+1}+\delta_{2M+1}\right]\tau_{L,j}$$

$$+\left[-\frac{5}{4}\,\bar{S}'_{13}\Big|_{2M+1}+\bar{\delta}_{2M+1}\right]\tau_0+\left\{\frac{-5h_j^2}{192}\,\bar{S}'_{33}\Big|_{2M+1}+\eta_{2M+1}\right\}\frac{d^2\tau_{L,j}}{dx^2}$$

$$+\left\{\frac{11h_j^2}{192}\,\bar{S}'_{33}\Big|_{2M+1}+\bar{\eta}_{2M+1}\right\}\frac{d^2\tau_0}{dx^2}$$

$$+\left[\bar{S}'_{13}\,\bar{Q}_{55}\Big|_{2M+1}+\mu_{2M+1}\right]\phi_j$$

$$+\Big\{-\bar{S}'_{33}\Big|_{2M+1}\left[\frac{13}{48}h_j^2\,\bar{Q}_{55}\Big|_{2M+1}-\frac{h_j}{2}\,\bar{\alpha}_{2M+1}\right]$$

$$+\rho_{2M+1}+\frac{h_jk_5}{2}\,\bar{S}'_{33}\Big|_{2M+1}+k_5\,\epsilon_{2M+1}\Big\}\frac{d^2\phi_j}{dx^2}\qquad\text{(over)}$$

$$+\left[\bar{S}'_{33}\Big|_{2M+1}\frac{h_j}{2}+\epsilon_{2M+1}\right]\frac{d\bar{v}_0}{dx}$$

$$+g_{2M+1}(x)+\alpha_z\Big|_{2M+1}\bar{g}(x,\tfrac{h_j}{2})+\beta_z\Big|_{2M+1}\bar{h}(x,\tfrac{h_j}{2}) \qquad (4.2.5)$$

Substituting equations (4.2.4) and (4.2.5) into equation (4.2.3) gives:

$$\frac{1}{C_N}\frac{d\bar{v}_0}{dx}=\frac{dw_i^0}{dx}+\left[-\frac{5}{4}\bar{S}'_{13}\Big|_1+\delta_1\right]\tau_0+\left[-\frac{1}{4}\bar{S}'_{13}\Big|_1+\bar{\delta}_1\right]\tau_{u,i}$$

$$+\left[\frac{11}{192}h_i^2\bar{S}'_{33}\Big|_1+\eta_1\right]\frac{d^2\tau_0}{dx^2}+\left[-\frac{5}{192}h_i^2\bar{S}'_{33}\Big|_1+\bar{\eta}_1\right]\frac{d^2\tau_{u,i}}{dx^2}$$

$$+\left[\bar{S}'_{13}\bar{Q}_{55}\Big|_1+\mu_1\right]\phi_i+\left[\frac{3}{48}h_i^2\bar{S}'_{33}\bar{Q}_{55}\Big|_1+\rho_1\right]\frac{d^2\phi_i}{dx^2}$$

$$+\left[-\frac{h_i}{2}\bar{S}'_{33}\Big|_1+\epsilon_1\right]\frac{d\bar{v}_0}{dx}+g_1(x)+\alpha_z\Big|_1\bar{g}(x,-\tfrac{h_i}{2})+\beta_z\Big|_1\bar{h}(x,-\tfrac{h_i}{2})$$

$$-\frac{dw_j^0}{dx}-\left[-\frac{1}{4}\bar{S}'_{13}\Big|_{2M+1}+\delta_{2M+1}\right]\tau_{L,j}-\left[-\frac{5}{4}\bar{S}'_{13}\Big|_{2M+1}+\bar{\delta}_{2M+1}\right]\tau_0$$

$$-\left[-\frac{5h_j^2}{192}\bar{S}'_{33}\Big|_{2M+1}+\eta_{2M+1}\right]\frac{d^2\tau_{L,j}}{dx^2}-\left[\frac{11h_j^2}{192}\bar{S}'_{33}\Big|_{2M+1}+\bar{\eta}_{2M+1}\right]\frac{d^2\tau_0}{dx^2}$$

$$-\left[\bar{S}'_{13}\bar{Q}_{55}\Big|_{2M+1}+\mu_{2M+1}\right]\phi_j-\Big\{-\bar{S}'_{33}\Big|_{2M+1}\Big[\frac{13}{48}h_j^2\bar{Q}_{55}\Big|_{2M+1}$$

$$-\frac{h_j}{2}\bar{\alpha}_{2M+1}\Big]+\rho_{2M+1}+\frac{h_jk_5}{2}\bar{S}'_{33}\Big|_{2M+1}+k_5\epsilon_{2M+1}\Big\}\frac{d^2\phi_j}{dx^2}$$

$$-\left[\frac{h_j}{2}\bar{S}'_{33}\Big|_{2M+1}+\epsilon_{2M+1}\right]\frac{d\bar{v}_0}{dx}-g_{2M+1}(x)-\alpha_z\Big|_{2M+1}\bar{g}(x,\tfrac{h_j}{2})$$

$$-\beta_z\Big|_{2M+1}\bar{h}(x,\tfrac{h_j}{2})$$

or, grouping terms for convenience,

$$0 = \frac{dw_i^o}{dx} - \frac{dw_j^o}{dx} + \gamma_1 \tau_o + \gamma_2 \frac{d^2\tau_o}{dx^2} + \gamma_3 \phi_i + \gamma_4 \phi_j + \gamma_5 \frac{d^2\phi_i}{dx^2} + \gamma_6 \frac{d^2\phi_j}{dx^2} + \gamma_7 \frac{d\sigma_o}{dx} + \gamma_8 \tau_{u,i} + \gamma_9 \frac{d^2\tau_{u,i}}{dx} + \gamma_{10} \tau_{L,j} + \gamma_{11} \frac{d^2\tau_{L,j}}{dx^2} + \gamma_{12}(x) \tag{4.2.6}$$

where

$$\gamma_1 = -\frac{5}{4} \bar{S}'_{13}\Big|_1 + \delta_1 + \frac{5}{4} \bar{S}'_{13}\Big|_{2M+1} - \bar{\delta}_{2M+1} \tag{4.2.7}$$

$$\gamma_2 = \frac{11}{192} h_i^2 \bar{S}'_{33}\Big|_1 - \eta_1 - \frac{11 h_j^2}{192} \bar{S}'_{33}\Big|_{2M+1} - \bar{\eta}_{2M+1} \tag{4.2.8}$$

$$\gamma_3 = \bar{S}'_{13} \bar{Q}_{55}\Big|_1 + \mu_1 \tag{4.2.9}$$

$$\gamma_4 = -\bar{S}'_{13} \bar{Q}_{55}\Big|_{2M+1} - \mu_{2M+1} \tag{4.2.10}$$

$$\gamma_5 = \frac{3}{48} h_i^2 \bar{S}'_{33} \bar{Q}_{55}\Big|_1 + \rho_1 \tag{4.2.11}$$

$$\gamma_6 = \bar{S}'_{33}\big|_{2M+1}\left[\frac{13}{48}h_j^2\,\bar{Q}_{55}\big|_{2M+1} - \frac{h_j}{2}\,\bar{\alpha}_{2M+1} - \frac{h_j k_5}{2}\right] - \rho_{2M+1} - k_5\,\epsilon_{2M+1} \tag{4.2.12}$$

$$\gamma_7 = -\frac{h_i}{2}\,\bar{S}'_{33}\big|_1 + \epsilon_1 - \frac{h_j}{2}\,\bar{S}'_{33}\big|_{2M+1} - \epsilon_{2M+1} - \frac{1}{C_N} \tag{4.2.13}$$

$$\gamma_8 = -\frac{1}{4}\,\bar{S}'_{13}\big|_1 + \bar{\delta}_1 \tag{4.2.14}$$

$$\gamma_9 = -\frac{5}{192}h_i^2\,\bar{S}'_{33}\big|_1 + \bar{\eta}_1 \tag{4.2.15}$$

$$\gamma_{10} = \frac{1}{4}\,\bar{S}'_{13}\big|_{2M+1} - \delta_{2M+1} \tag{4.2.16}$$

$$\gamma_{11} = \frac{5}{192}h_j^2\,\bar{S}'_{33}\big|_{2M+1} - \eta_{2M+1} \tag{4.2.17}$$

$$\gamma_{12}(x) = g_1(x) + \alpha_z\big|_1\,\bar{g}\left(x,-\frac{h_i}{2}\right) + \beta_z\big|_1\,\bar{h}\left(x,-\frac{h_i}{2}\right) - g_{2M+1}(x) - \alpha_z\big|_{2M+1}\,\bar{g}\left(x,\frac{h_j}{2}\right) - \beta_z\big|_{2M+1}\,\bar{h}\left(x,\frac{h_j}{2}\right) \tag{4.2.18}$$

We will now perform similar operations on the equations for u to obtain the equation (4.2.2) in terms of our unknown functions. Using equation (2.4.67)

$$u_i(x,z)\Big|_{\substack{z=-h_i/2 \\ k=1}} \equiv u_i\left(x,-\frac{h_i}{2}\right)$$

$$
\begin{aligned}
u_i(x,{}^{-h_i}\!/_2) = {} & u_i^{\circ} - \left(-\frac{h_i}{2}\right)\frac{dw_i^{\circ}}{dx} + \bar{\bar{h}}_1(x,{}^{-h_i}\!/_2) \\
& + \Bigg[\bar{S}_{13}'\Big|_1 \left(\frac{-(-h_i/2)^2}{2h_i} + \frac{(-h_i/2)^3}{h_i^2}\right) + (\Delta + \bar{S}_{55})_1 \cdot \\
& \left(-\frac{(-h_i/2)}{2} - \frac{(-h_i/2)^2}{h_i} + \frac{2(-h_i/2)^3}{h_i^2}\right) + G_1 - \delta_1\left({}^{-h_i}\!/_2\right)\Bigg]\tau_0 \\
& + \Bigg[\bar{S}_{13}'\Big|_1 \left(\frac{(-h_i/2)^2}{2h_i} + \frac{(-h_i/2)^3}{h_i^2}\right) + (\Delta + \bar{S}_{55})_1 \cdot \\
& \left(-\frac{(-h_i/2)}{2} + \frac{(-h_i/2)^2}{h_i} + 2\frac{(-h_i/2)^3}{h_i^2}\right) + \bar{G}_1 - \bar{\delta}_1\left(\frac{-h_i}{2}\right)\Bigg]\tau_{u,i} \\
& + \Bigg[\bar{S}_{33}'\Big|_1 \left(-\frac{(-h_i/2)^3}{24} - \frac{(-h_i/2)^4}{24h_i} + \frac{(-h_i/2)^5}{20h_i^2} + \frac{h_i(-h_i/2)^2}{16}\right) \\
& + F_1 - \eta_1(-h_i/2)\Bigg]\frac{d^2\tau_0}{dx^2} \\
& + \Bigg[\bar{S}_{33}'\Big|_1 \left(-\frac{(-h_i/2)^3}{24} + \frac{(-h_i/2)^4}{24h_i} + \frac{(-h_i/2)^5}{20h_i^2} - \frac{h_i(-h_i/2)^2}{16}\right) \\
& + \bar{F}_1 - \bar{\eta}_1(-h_i/2)\Bigg]\frac{d^2\tau_{u,i}}{dx^2} \\
& + \Bigg[(2 + 2\Delta\bar{Q}_{55})_1\left(-h_i/2 - \frac{4(-h_i/2)^3}{3h_i^2}\right) + \cancelto{0}{(2\Delta + 2\bar{S}_{55})b_{55}\Big|_1}(-h_i/2) \\
& - \frac{4\bar{S}_{13}'\bar{Q}_{55}\Big|_1}{3h_i^2}(-h_i/2)^3 + E_1 - \mu_1(-h_i/2)\Bigg]\phi_i \qquad \text{(over)}
\end{aligned}
$$

$$+\left\{ \bar{S}'_{33}\Big|_1 \left[\bar{Q}_{55}\Big|_1 \left(\frac{(-h_i/2)^3}{6} - \frac{(-h_i/2)^5}{15h_i^2} \right) - \cancelto{0}{\bar{\alpha}_1} \frac{(-h_i/2)^2}{2} \right.\right.$$

$$\left.\left. + \bar{Q}_{55}\Big|_1 \frac{h_i(-h_i/2)^2}{6} + \cancelto{0}{b_{53}\Big|_1} \frac{(-h_i/2)^3}{6} \right] + D_1 - \rho_1\left(-\frac{h_i}{2}\right) \right\} \frac{d^2\phi_i}{dx^2}$$

$$+\left[-\bar{S}'_{33}\Big|_1 \frac{(-h_i/2)^2}{2} + H'_1 - \epsilon_1\left(-\frac{h_i}{2}\right) \right] \frac{d\tau_0}{dx}$$

or,

$$u_i(x,-h_i/2) = u_i^0 + \frac{h_i}{2}\frac{dw_i^0}{dx} + \bar{\bar{h}}_1(x,-\frac{h_i}{2}) + \left[-\frac{h_i}{4}\bar{S}'_{13}\Big|_1 \right.$$

$$\left. + G_1 + \frac{h_i}{2}\delta_1 \right] \tau_0$$

$$+\left[(\Delta + \bar{S}_{55})_1 \cdot \left(\frac{h_i}{4}\right) + \bar{G}_1 + \frac{h_i}{2}\bar{\delta}_1 \right] \tau_{u,i}$$

$$+\left[\frac{1}{60} h_i^3 \bar{S}'_{33}\Big|_1 + F_1 + \frac{h_i}{2}\eta_1 \right] \frac{d^2\tau_0}{dx^2}$$

$$+\left[\frac{-3}{320} h_i^3 \bar{S}'_{33}\Big|_1 + \bar{F}_1 + \frac{h_i}{2}\bar{\eta}_1 \right] \frac{d^2\tau_{u,i}}{dx^2}$$

$$+\left[(2 + 2\Delta\bar{Q}_{55})_1 \left(-\frac{h_i}{3}\right) + \frac{h_i}{6}\bar{S}'_{13}\bar{Q}_{55}\Big|_1 + E_1 + \frac{h_i}{2}\mu_1 \right] \phi_i$$

$$+\left\{ \bar{S}'_{33}\Big|_1 \left[\bar{Q}_{55}\Big|_1 \left(\frac{11}{480}h_i^3\right) \right] + D_1 + \frac{h_i}{2}\rho_1 \right\} \frac{d^2\phi_i}{dx^2}$$

$$+\left[-\frac{h_i^2}{8}\bar{S}'_{33}\Big|_1 + H'_1 + \frac{h_i}{2}\epsilon_1 \right] \frac{d\tau_0}{dx} \qquad (4.2.19)$$

Next, use equation (4.1.1) with $k = 2M + 1$, $z = h_i/2$.
We use symmetry, $\bar{Q}_{55}\big|_1 = \bar{Q}_{55}\big|_{2M+1}$, in the term multiplying $\frac{d^2\phi_i}{dx^2}$ to avoid confusion.

$$
\begin{aligned}
u_j(x, h_j/2) = {} & u_j^{\circ} - (h_j/2)\frac{dw_j^{\circ}}{dx} + \bar{\bar{h}}_{2M+1}(x, h_j/2) \\
& + \Bigg[\bar{S}'_{13}\Big|_{2M+1}\left(-\frac{(h_j/2)^2}{2h_j} + \frac{(h_j/2)^3}{h_j^2} \right) + (\Delta + \bar{S}_{55})_{2M+1} \cdot \\
& \quad \left(-\frac{(h_j/2)}{2} - \frac{(h_j/2)^2}{h_j} + \frac{2(h_j/2)^3}{h_j^2} \right) + G_{2M+1} - \delta_{2M+1}(h_j/2) \Bigg] \tau_{L,j} \\
& + \Bigg[\bar{S}'_{13}\Big|_{2M+1}\left(\frac{(h_j/2)^2}{2h_j} + \frac{(h_j/2)^3}{h_j^2} \right) + (\Delta + \bar{S}_{55})_{2M+1} \cdot \\
& \quad \left(-\frac{(h_j/2)}{2} + \frac{(h_j/2)^2}{h_j} + \frac{2(h_j/2)^3}{h_j^2} \right) + \bar{G}_{2M+1} - \bar{\delta}_{2M+1}(h_j/2) \Bigg] \tau_{0} \\
& + \Bigg[\bar{S}'_{33}\Big|_{2M+1}\left(-\frac{(h_j/2)^3}{24} - \frac{(h_j/2)^4}{24h_j} + \frac{(h_j/2)^5}{20h_j^2} + \frac{h_j(h_j/2)^2}{16} \right) \\
& \quad + F_{2M+1} - \eta_{2M+1}(h_j/2) \Bigg] \frac{d^2\tau_{L,j}}{dx^2} \\
& + \Bigg[\bar{S}'_{33}\Big|_{2M+1}\left(-\frac{(h_j/2)^3}{24} + \frac{(h_j/2)^4}{24h_j} + \frac{(h_j/2)^5}{20h_j^2} - \frac{h_j(h_j/2)^2}{16} \right) \\
& \quad + \bar{F}_{2M+1} - \bar{\eta}_{2M+1}(h_j/2) \Bigg] \frac{d^2\tau_{0}}{dx^2} \\
& + \Bigg[(2 + 2\Delta\bar{Q}_{55})_{2M+1}\left(h_j/2 - \frac{4(h_j/2)^3}{3h_j^2} \right) \\
& \quad + (2\Delta + 2\bar{S}_{55})_{2M+1}\, \cancelto{0}{b_{55}\big|_{2M+1}}\,(h_j/2) \\
& \quad - \frac{4\bar{S}'_{13}\bar{Q}_{55}\big|_{2M+1}}{3h_j^2}(h_j/2)^3 + E_{2M+1} - \mu_{2M+1}(h_j/2) \Bigg] \phi_j \\
& + \Bigg\{ \bar{S}'_{33}\Big|_{2M+1}\Bigg[\bar{Q}_{55}\Big|_{2M+1}\left(\frac{(h_j/2)^3}{6} - \frac{(h_j/2)^5}{15h_j^2} \right) - \cancelto{0}{\alpha_1}\frac{(h_j/2)^2}{2} \\
& \quad + \bar{Q}_{55}\Big|_{2M+1}\frac{h_j(h_j/2)^2}{6} + \cancelto{0}{b_{55}}\Big|_{2M+1}\frac{(h_j/2)^3}{6} \Bigg] \\
& \quad + D_{2M+1} - \rho_{2M+1}(h_j/2) - k_5\bar{S}'_{33}\Big|_{2M+1}\frac{(h_j/2)^2}{2}
\end{aligned}
$$

(over)

$$+ k_s H'_{2M+1} - k_s \epsilon_{2M+1}(h_j/2) \Big\} \frac{d^2\phi_j}{dx^2}$$

$$+ \left[-\bar{S}'_{33}\Big|_{2M+1} \frac{(h_j/2)^2}{2} + H'_{2M+1} - \epsilon_{2M+1}(h_j/2) \right] \frac{d\,\sigma_0}{dx}$$

or,

$$u_j(x, h_j/2) = u_j^\circ - \frac{h_j}{2}\frac{dw_j^\circ}{dx} + \bar{\bar{h}}_{2M+1}(x, h_j/2)$$

$$+ \Big[(\Delta + \bar{S}_{55})_{2M+1}\left(-\frac{h_j}{4}\right)$$

$$+ G_{2M+1} - \frac{h_j}{2}\delta_{2M+1} \Big] \tau_{L,j} + \Big[\bar{S}'_{13}\Big|_{2M+1}\left(\frac{h_j}{4}\right)$$

$$+ (\Delta + \bar{S}_{55})_{2M+1}\left(\frac{h_j}{4}\right) + \bar{G}_{2M+1} - \frac{h_j}{2}\bar{\delta}_{2M+1} \Big] \tau_0$$

$$+ \left[\bar{S}'_{33}\Big|_{2M+1}\left(\frac{3\,h_j^3}{320}\right) + F_{2M+1} - \frac{h_j}{2}\eta_{2M+1} \right] \frac{d^2\tau_{L,j}}{dx^2}$$

$$+ \left[\bar{S}'_{33}\Big|_{2M+1}\left(-\frac{h_j^3}{60}\right) + \bar{F}_{2M+1} - \frac{h_j}{2}\bar{\eta}_{2M+1} \right] \frac{d^2\tau_0}{dx^2}$$

$$+ \Big[(2 + 2\Delta\bar{Q}_{55})_{2M+1}\left(\frac{h_j}{3}\right) - \frac{h_j}{6}\bar{S}'_{13}\bar{Q}_{55}\Big|_{2M+1}$$

$$+ E_{2M+1} - \frac{h_j}{2}\mu_{2M+1} \Big] \phi_j$$

$$+ \Big\{ \bar{S}'_{33}\Big|_{2M+1}\left[\bar{Q}_{55}\Big|_{2M+1}\left(\frac{29}{480}h_j^3\right) \right] + D_{2M+1} - \frac{h_j}{2}\rho_{2M+1}$$

$$- \frac{h_j^2}{8} k_s \bar{S}'_{33}\Big|_{2M+1} + k_s H'_{2M+1} - k_s \frac{h_j}{2}\epsilon_{2M+1} \Big\} \frac{d^2\phi_j}{dx^2}$$

$$+ \left[-\frac{h_j^2}{8}\bar{S}'_{33}\Big|_{2M+1} + H'_{2M+1} - \frac{h_j}{2}\epsilon_{2M+1} \right] \frac{d\,\sigma_0}{dx} \qquad (4.2.20)$$

Combining equations (4.2.19) and (4.2.20) with equation (4.2.2) gives

$$
\begin{aligned}
\frac{\tau_0}{C_s} = {} & u_i^0 + \frac{h_i}{2}\frac{dw_i^0}{dx} + \bar{\bar{h}}_1\left(x, -\frac{h_i}{2}\right) + \left[-\frac{h_i}{4}\bar{S}'_{13}\Big|_1 + G_1 + \frac{h_i}{2}\delta_1 \right]\tau_0 \\
& + \left[\frac{h_i}{4}(\Delta + \bar{S}_{55})_1 + \bar{G}_1 + \frac{h_i}{2}\bar{\delta}_1 \right]\tau_{u,i} \\
& + \left[\frac{h_i^3}{60}\bar{S}'_{33}\Big|_1 + F_1 + \frac{h_i}{2}\eta_1 \right]\frac{d^2\tau_0}{dx^2} \\
& + \left[\frac{-3}{320}h_i^3\bar{S}'_{33}\Big|_1 + \bar{F}_1 + \frac{h_i}{2}\bar{\eta}_1 \right]\frac{d^2\tau_{u,i}}{dx^2} \\
& + \left[-\frac{h_i}{3}(2+2\Delta\bar{Q}_{55})_1 + \frac{h_i}{6}\bar{S}'_{13}\bar{Q}_{55}\Big|_1 + E_1 + \frac{h_i}{2}\mu_1 \right]\phi_i \\
& + \left[\frac{11}{480}h_i^3\bar{S}'_{33}\bar{Q}_{55}\Big|_1 + D_1 + \frac{h_i}{2}\rho_1 \right]\frac{d^2\phi_i}{dx^2} \\
& + \left[\frac{-h_i^2}{8}\bar{S}'_{33}\Big|_1 + H'_1 + \frac{h_i}{2}\epsilon_1 \right]\frac{dv_0}{dx} \\
& - u_j^0 + \frac{h_j}{2}\frac{dw_j^0}{dx} - \bar{\bar{h}}_{2M+1}\left(x, h_j/2\right) \\
& - \left[-\frac{h_j}{4}(\Delta + \bar{S}_{55})_{2M+1} + G_{2M+1} - \frac{h_j}{2}\delta_{2M+1} \right]\tau_{L,j} \\
& - \left[\frac{h_j}{4}\bar{S}'_{13}\Big|_{2M+1} + \frac{h_j}{4}(\Delta + \bar{S}_{55})_{2M+1} + \bar{G}_{2M+1} - \frac{h_j}{2}\bar{\delta}_{2M+1} \right]\tau_0 \\
& - \left[\frac{3h_j^3}{320}\bar{S}'_{33}\Big|_{2M+1} + F_{2M+1} - \frac{h_j}{2}\eta_{2M+1} \right]\frac{d^2\tau_{L,j}}{dx^2} \\
& - \left[-\frac{h_j^3}{60}\bar{S}'_{33}\Big|_{2M+1} + \bar{F}_{2M+1} - \frac{h_j}{2}\bar{\eta}_{2M+1} \right]\frac{d^2\tau_0}{dx^2} \\
& - \left[\frac{h_j}{3}(2+2\Delta\bar{Q}_{55})_{2M+1} - \frac{h_j}{6}\bar{S}'_{13}\bar{Q}_{55}\Big|_{2M+1} + E_{2M+1} - \frac{h_j}{2}\mu_{2M+1} \right]\phi_j
\end{aligned}
$$

(over)

$$-\left[\frac{29 h_i^3}{480}\bar{S}_{33}'\bar{Q}_{55}\Big|_{2M+1} + D_{2M+1} - \frac{h_j}{2}\rho_{2M+1} - \frac{h_j^2}{8}k_s\bar{S}_{33}'\Big|_{2M+1}\right.$$

$$\left. + k_s H_{2M+1}' - \frac{h_j}{2}k_s\epsilon_{2M+1}\right]\frac{d^2\phi_j}{dx^2}$$

$$-\left[-\frac{h_j^2}{8}\bar{S}_{33}'\Big|_{2M+1} + H_{2M+1}' - \frac{h_j}{2}\epsilon_{2M+1}\right]\frac{d\sigma_0}{dx}$$

or, grouping terms for convenience,

$$0 = u_i^0 - u_j^0 + \frac{h_i}{2}\frac{dw_i^0}{dx} + \frac{h_j}{2}\frac{dw_j^0}{dx}$$

$$+\bar{\gamma}_1\tau_0 + \bar{\gamma}_2\frac{d^2\tau_0}{dx^2} + \bar{\gamma}_3\phi_i + \bar{\gamma}_4\phi_j$$

$$+\bar{\gamma}_5\frac{d^2\phi_i}{dx^2} + \bar{\gamma}_6\frac{d^2\phi_j}{dx^2} + \bar{\gamma}_7\frac{d\sigma_0}{dx} + \bar{\gamma}_8\tau_{u,i}$$

$$+\bar{\gamma}_9\frac{d^2\tau_{u,i}}{dx^2} + \bar{\gamma}_{10}\tau_{L,j} + \bar{\gamma}_{11}\frac{d^2\tau_{L,j}}{dx^2} + \bar{\gamma}_{12}(x) \qquad (4.2.21)$$

where

$$\bar{\gamma}_1 = -\frac{h_i}{4}\bar{S}_{13}'\Big|_1 + G_1 + \frac{h_i}{2}\delta_1 - \frac{h_j}{4}\bar{S}_{13}'\Big|_{2M+1}$$

$$-\frac{h_j}{4}(\Delta + \bar{S}_{55})_{2M+1} - \bar{G}_{2M+1} + \frac{h_j}{2}\bar{\delta}_{2M+1} - \frac{1}{C_s} \qquad (4.2.22)$$

$$\bar{\gamma}_2 = \frac{h_i^3}{60}\bar{S}_{33}'\Big|_1 + F_1 + \frac{h_i}{2}\eta_1 + \frac{h_j^3}{60}\bar{S}_{33}'\Big|_{2M+1}$$

$$-\bar{F}_{2M+1} + \frac{h_j}{2}\bar{\eta}_{2M+1} \qquad (4.2.23)$$

$$\bar{\gamma}_3 = -\frac{h_i}{3}(2+2\Delta\bar{Q}_{55})_1 + \frac{h_i}{6}\bar{S}_{13}'\bar{Q}_{55}\big|_1 + E_1 + \frac{h_i}{2}\mu_1 \tag{4.2.24}$$

$$\bar{\gamma}_4 = \frac{-h_j}{3}(2+2\Delta\bar{Q}_{55})_{2M+1} + \frac{h_j}{6}\bar{S}_{13}'\bar{Q}_{55}\big|_{2M+1} - E_{2M+1} + \frac{h_j}{2}\mu_{2M+1} \tag{4.2.25}$$

$$\bar{\gamma}_5 = \frac{11}{480}h_i^3\,\bar{S}_{33}'\bar{Q}_{55}\big|_1 + D_1 + \frac{h_i}{2}\rho_1 \tag{4.2.26}$$

$$\bar{\gamma}_6 = -\frac{29}{480}h_j^3\,\bar{S}_{33}'\bar{Q}_{55}\big|_{2M+1} - D_{2M+1} + \frac{h_j}{2}\rho_{2M+1} + \frac{h_j^2}{8}k_5\bar{S}_{33}'\big|_{2M+1} - k_5 H_{2M+1}' + \frac{h_j}{2}k_5\epsilon_{2M+1} \tag{4.2.27}$$

$$\bar{\gamma}_7 = -\frac{h_i^2}{8}\bar{S}_{33}'\big|_1 + H_1' + \frac{h_i}{2}\epsilon_1 + \frac{h_j^2}{8}\bar{S}_{33}'\big|_{2M+1} - H_{2M+1}' + \frac{h_j}{2}\epsilon_{2M+1} \tag{4.2.28}$$

$$\bar{\gamma}_8 = \frac{h_i}{4}(\Delta+\bar{S}_{55})_1 + \bar{G}_1 + \frac{h_i}{2}\bar{\delta}_1 \tag{4.2.29}$$

$$\bar{\gamma}_9 = \frac{-3}{320}h_i^3\,\bar{S}_{33}'\big|_1 + \bar{F}_1 + \frac{h_i}{2}\bar{\eta}_1 \tag{4.2.30}$$

$$\bar{\gamma}_{10} = \frac{h_j}{4}\left(\Delta + \bar{S}_{55}\right)_{2M+1} - G_{2M+1} + \frac{h_j}{2}\,\delta_{2M+1} \tag{4.2.31}$$

$$\bar{\gamma}_{11} = -\frac{3h_j^3}{320}\,\bar{S}'_{33}\Big)_{2M+1} - F_{2M+1} + \frac{h_j}{2}\,\eta_{2M+1} \tag{4.2.32}$$

$$\bar{\gamma}_{12}(x) = \bar{\bar{h}}_1(x, -h_j/2) - \bar{\bar{h}}_{2M+1}(x, h_j/2) \tag{4.2.33}$$

4.3 Discussion of Bonded Joint Equations

We now summarize the unknown functions in a bonded joint. For elements "i" and "j" , (Fig. 15) we have 14 unknown functions.

$$\phi_i,\ \phi_j,\ u_i^0,\ u_j^0,\ w_i^0,\ w_j^0,\ N_i,\ N_j,\ M_i,\ M_j,\ Q_i,\ Q_j,\ \tau_0,\ \sigma_0$$

and six known functions

$$N^T,\ N^m,\ M^T,\ M^m,\ \tau_{u,i},\ \tau_{L,j}$$

The equations for this system are:

$$\left.\begin{array}{ll} 3 & \text{equilibrium, element } i \\ 3 & \text{equilibrium, element } j \end{array}\right\} (2.6.1,\ 2.6.2,\ 2.6.3)$$

$$\left.\begin{array}{ll} 3 & \text{definitions of } M_i,\ N_i,\ Q_i \\ 3 & \text{definitions of } M_j,\ N_j,\ Q_j \end{array}\right\} (2.5.6,\ 2.5.18\ \ 2.5.31)$$

$$\begin{array}{ll} \underline{2} & \text{interface equations } (4.2.6,\ 4.2.21) \\ 14 & \text{total} \end{array}$$

W.J. Renton [Ref. 8] has found that the reduction may be accomplished by obtaining two linear, constant co-efficient, ordinary differential equations in the two unknowns τ_0 and Q_i. In fact, either Q_i or Q_j may be used, since, by the ΣF_y equilibrium equations for the adherend elements i and j , (See Figs. 15 and 16)

$$0 = \frac{dQ_i}{dx} - \sigma_0 + \sigma_{u,i}$$

and

$$0 = \frac{dQ_j}{dx} - \sigma_{L,j} + \sigma_0$$

yielding

$$\frac{dQ_i}{dx} = - \frac{dQ_j}{dx} + \sigma_{L,j} - \sigma_{u,i}$$

If the adherends "i" and "j" are identical, the equations uncouple. For dissimilar adherends, using operator multiplication will reduce the system to one equation in one unknown, an eighth order ordinary differential equation in τ_0 .

Unknown constants in the system can be evaluated as followed: 3 unique constants for each element (match either u^0, w^0, and $\frac{dw^0}{dx}$, or, preferably, M, N, and Q as boundary conditions at each end of the element), and τ_0 (x=0) = τ_0 (x=L) = 0 where L = length (x distance) of the interface. This last provision is necessary to have τ_0 be zero on each end of the bond where there is a free surface. The main difficulty in the solution is algebraic, since we must derive equations involving τ_0 and each of the other quantities of interest so that we may use the boundary conditions on the elements i and j to solve for the unknown coefficients in τ_0 .

CHAPTER 5

Conclusions

An analytical model has been developed for the plane-strain analysis of symmetric anisotropic laminated plates subjected to a full range of thermal, moisture, and mechanical loadings including surface tractions. This analysis includes transverse shear strain ε_{xz} and transverse normal strain ε_z, and can be used for any laminate which through engineering judgement can be assumed to be in plane-strain in the plane of loading. The methods of analysis have been demonstrated to be accurate for a single laminated plate. The example problems (section 3.2) show the method to give results which are consistent with physical intuition. In addition, problems of current interest, including residual curing stresses and hygrothermal loadings (combined elevated temperature and moisture) have been examined.

The edge loads on the y = constant edges arise as a natural consequence of the plane-strain assumption, and the displacement "v" in the y direction is zero. Should the displacements or loadings in the y direction

become important, then the plane-strain assumption herein will no longer be valid.

The results of calculations involving the tracing constants a, b, and c show that their effects are important. The difference in material constants is greatest in $\bar{S}'_{13}$ (the stretching-shearing compliance term corrected for off-axis effects, equation 2.4.12a); there is a factor of two difference in $\bar{S}'_{13}$ depending on whether $a = 0$ or $a = 1$. For constants $a=b=c=1$ or $a=b=c=0$, the constant "C" (multiplying $\frac{d^3\tau_L}{dx^3}$ in equation 2.5.18 for N) varies by a factor of five. There are many other constants, however, which do not vary appreciably with the values of a, b, and c. Thus, whether a, b, and c are one or zero, the stress and displacement calculations only vary by about 3 to 4%. To take advantage of algebraic completeness, constants a, b, and c should always be set equal to unity; the computer program in Appendix B has these constants set equal to one as default values.

There are many potential uses of this plate element as an "analytic finite element". While the usage of aggregations of these plate elements is only suggested in Chapter 4, the single lap joint (see Fig. 3) as well as the double lap or step-lap joint may be analyzed by the methods of analysis developed herein. The only difficulty

with such solutions is the length of the algebraic manipulations needed; there is no difficulty with the concept of using the plate as a finite element for the constructions discussed above.

References

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APPENDIX

Comparison of this analysis with previous analysis by W.J. Renton and J.R. Vinson as presented in AFOSR TR 75-0125, Aug. 1974

I. Use of Tracing Constants a, b, c

Several terms were omitted in Renton's analysis [Ref. 8] but have been included in this report. The work in this report is algebraically complete, and the tracing terms should all be set equal to 1 to take advantage of this fact. All tracing constants would be set equal to zero to give the Renton results. This difference will affect most calculations <5% . See equations (2.4.12), (2.4.26), and (2.5.1c) for these constants.

II. Plane-Strain Assumption

Renton assumed that

$$(\varepsilon_y - \alpha_y \Delta T)_k = 0$$

$$(\varepsilon_{yz})_k = 0$$

$$(\varepsilon_{xy} - \alpha_{xy} \Delta T)_k = 0$$

There is some difficulty with this form of assumption, however. For example, with a constant ΔT, and dissimilar layers i and j, we have $(\alpha_y)_i \neq (\alpha_y)$; If, as stated by Renton

$$(\varepsilon_y)_k = \left(\frac{\partial v}{\partial y} \right)_k = (\alpha_y) \; \Delta T \qquad k = i,j$$

then the layers will have, as y varies,

$$v_i \neq v_j$$

at the interface. The layers will thus slide freely over each other and delaminate.

The assumptions in analysis herein are those of a classical plane-strain condition, including moisture effects in the constitutive equations:

$$(\varepsilon_y)_k = 0$$

$$(\varepsilon_{yz})_k = 0$$

$$(\varepsilon_{xy})_k = 0$$

III. Inclusion of Moisture Loadings

Renton's analysis makes allowance for thermal strains only, with constant thermal expansion coefficients. This report includes thermal strains with constant thermal expansion coefficients and also moisture strains with constant moisture expansion coefficients.